Notes on C^*-algebras

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PREFACE

These are the lecture notes prepared for a course on C^*-algebras given by the author at IMSC, Chennai during Sept 2019-March 2020. The targeted audience were graduate students interested in working in the area of operator algebras. The aim of this course was to make the audience familiar with few basic notions in the theory of C^*-algebras and make them familiar with the language required to read the current literature on the subject. The topics discussed in this notes (we merely scratch the surface as the motivation is to make the reader converse in the language as opposed to giving a complete treatment) are universal C^*-algebras, group C^*-algebras, crossed products, Hilbert C^*-modules, Morita equivalence, and K-theory.

The prospective reader of these notes is assumed to have acquaintance with the following topics in C^*-algebras.

(1) Gelfand-Naimark theorem for commutative C^*-algebras,

(2) continuous functional calculus,

(3) the notion of positivity, states and the GNS construction,

(4) the quotient construction in C^*-algebras, and

(5) the existence of approximate identities in C^*-algebras.

Arveson's "Invitation to C^*-algebras" is an ideal and a highly recommended book to learn the above mentioned topics. The organisation of this notes is as follows.

In the first few sections, we discuss a few examples of C^*-algebras. The first example we discuss is the algebra of compact operators and realise them as a universal C^*-algebra given in terms of generators and relations. This serves as a model for the notion of universal C^*-algebras. The universal C^*-algebras allows us to quickly define group C^*-algebras and crossed products, two classes of examples extensively studied in the literature. Several important C^*-algebras studied in the literature has this universal prescription so it is appropriate to give a rigorous treatment. A little glimpse to the world of semigroup C^*-algebras is provided with a treatment of the Toeplitz algebra. After quickly reviewing the measure theoretic preliminaries in Section 4, we discuss in Section 5 group C^*-algebras associated to a locally compact second countable Hausdorff group.

In Section 6, we take up crossed products of C^*-algebras. After defining the full and reduced crossed product, we introduce Hilbert C^*-modules in Section 7 as a tool to prove that the reduced C^*-norm is independent of the choice of the representation that one chooses. The author believes that it is an appropriate point to introduce the notion of Hilbert C^*-modules to the reader. As an application of the machinery of crossed products, Stone-von Neumann theorem regarding the uniqueness of irreducible Weyl representations is proved in Section 8. After a short discussion on the non-commutative torus, we discuss Rieffel's proof of Mackey's imprimitivity theorem in Section 10. The notion of Morita equivalence is introduced and a proof is provided in the discrete setting.

Sections 11-16 in itself constitute a short course on K-theory. After deriving the basic properties of K_0 and K_1, the chapter culminates with the proof of Bott periodicity due to Cuntz. The treatment on K-theory, and also on universal C^*-algebras is based on three lectures given by Cuntz during the Oberwolfach conference on semigroup C^*-algebras held in Oct 2014. It also borrows material from the Master's thesis of Prakash Kumar Singh, a former student of CMI, Chennai , done under the author's supervision.

There are several excellent resources to read about the material covered in this notes. The bibliography contains a sample list. It is certainly not exhaustive and I apologise sincerely for any omission. The author claims no originality for the material presented nor for the way it is presented.

I would to like end this short introduction by thanking a few people who have helped me immensely so far. First, I would like to thank V. S. Sunder and my advisor Partha Sarathi Chakraborty for teaching me several aspects of mathematics which has enriched my understanding of the subject. I thank Arup, Bipul and Prakash for several discussions on K-theory. I thank Anbu and Murugan for discussions regarding the uniqueness of Weyl relations. Last but not least, I thank the participants of this course Sruthy, Piyasa and Jayakumar for attending all the lectures and their enthusiasm shown which kept me going for the entire length.

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1 The C^*-algebra of compact operators

The first C^*-algebra of interest is the algebra of compact operators on a separable Hilbert space. We assume throughout that all the Hilbert spaces that we consider are separable. Our convention is that the inner product is linear in the first variable and antilinear in the second variable. Let us recall the following facts usually learnt in a first course on functional analysis. Let $\mathcal{H}$ be a separable Hilbert space.

(1) A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be compact if the following condition is satisfied. Suppose (x_n) is a bounded sequence in $\mathcal{H}$. Then (Tx_n) has a convergent subsequence in $\mathcal{H}$.

(2) Denote the set of compact operators on $\mathcal{H}$ by $\mathcal{K}(\mathcal{H})$. Then $\mathcal{K}(\mathcal{H})$ is a norm closed two sided ideal in $B(\mathcal{H})$. Moreover $\mathcal{K}(\mathcal{H})$ is closed under taking adjoints.

(3) An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be finite rank if $Ran(T)$ is finite dimensional. Denote the set of finite rank operators on $\mathcal{H}$ by $\mathcal{F}(\mathcal{H})$. Then $\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{K}(\mathcal{H})$.

For $\xi, \eta \in \mathcal{H}$, let $\theta_{\xi,\eta} \in B(\mathcal{H})$ be defined by $\theta_{\xi,\eta}(\gamma) = \xi\langle\gamma|\eta\rangle$. Clearly, $\theta_{\xi,\eta}$ is of rank one and hence compact. Note the following relations.

$$\begin{aligned}
\theta^*_{\xi,\eta} &= \theta_{\eta,\xi} \\
\theta_{\xi_1,\eta_1}\theta_{\xi_2,\eta_2} &= \langle\xi_2|\eta_1\rangle\theta_{\xi_1,\eta_2} \\
T\theta_{\xi,\eta} &= \theta_{T\xi,\eta} \\
\theta_{\xi,\eta}T &= \theta_{\xi,T^*\eta}
\end{aligned}$$

for $\xi, \eta, \xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}$ and $T \in B(\mathcal{H})$. The above relations imply that the linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in \mathcal{H}\}$ is a $*$-closed subalgebra of $\mathcal{K}(\mathcal{H})$.

Exercise 1.1 *Prove that the linear span of* $\{\theta_{\xi,\eta} : \xi, \eta \in \mathcal{H}\}$ *is* $\mathcal{F}(\mathcal{H})$.

Observe that the map $\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \to \theta_{\xi,\eta} \in \mathcal{K}(\mathcal{H})$ is linear in the first variable and antilinear in the second variable. Note that for $\xi, \eta \in \mathcal{H}$, $||\theta_{\xi,\eta}|| = ||\xi||||\eta||$. These two facts and the fact that $\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{K}(\mathcal{H})$ together imply that if D is a countable dense subset of $\mathcal{H}$, then $\{\theta_{\xi,\eta} : \xi, \eta \in D\}$ is total in $\mathcal{K}(\mathcal{H})$. Thus $\mathcal{K}(\mathcal{H})$ is a separable C^*-subalgebra of $B(\mathcal{H})$.

The two most important results regarding the C^*-algebra of compacts is that $\mathcal{K}(\mathcal{H})$ is simple, i.e. it has no nontrivial two sided ideals and that $\mathcal{K}(\mathcal{H})$ has only one irreducible representation up to unitary equivalence.

Theorem 1.1 *Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{K}(\mathcal{H})$ is simple.*

First we prove the result assuming $\mathcal{H}$ is finite dimensional. Suppose $\dim(\mathcal{H}) = n$. Then every linear operator on $\mathcal{H}$ is compact and consequently $\mathcal{K}(\mathcal{H})$ is isomorphic to $M_n(\mathbb{C})$.

Lemma 1.2 *For $n \geq 1$, $M_n(\mathbb{C})$ is simple.*

Proof. Let $n \geq 1$ be given. For $i, j \in \{1, 2, \cdots, n\}$, let e_{ij} be the matrix with 1 at the $(i, j)^{th}$-entry and zero everywhere. Then $\{e_{ij}\}_{i,j}$ forms a basis for $M_n(\mathbb{C})$. Note the following relations.

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$
$$e_{ij}^* = e_{ji}.$$

for $i, j, k, l \in \{1, 2, \cdots, n\}$. Let I be a non-zero two sided ideal in $M_n(\mathbb{C})$. Pick a non-zero element $X \in I$. Write $X = \sum_{i,j} x_{ij}e_{ij}$. There exists k, l such that $x_{kl} \neq 0$. Note that $e_{kk}Xe_{ll} = x_{kl}e_{kl}$. Hence $e_{kl} \in I$ as I is a two sided ideal. Let $i, j \in \{1, 2, \cdots, n\}$ be given. Note that $e_{ij} = e_{ik}e_{kl}e_{lj}$. Hence $e_{ij} \in I$ for every i, j. But $\{e_{ij}\}_{i,j}$ is a basis for $M_n(\mathbb{C})$. As a consequence, it follows that $I = M_n(\mathbb{C})$. This completes the proof. □

Lemma 1.3 *Let A be a C^*-algebra. The following are equivalent.*

(1) For every non-zero representation π, $||\pi(a)|| = ||a||$.

(2) The C^*-algebra A is simple.

Proof. Suppose (1) holds. Let I be a non-zero ideal in A. Let $\pi : A/I \to B(\mathcal{H}_\pi)$ be a faithful representation of A/I. Denote the quotient map $A \to A/I$ by q. Then for every $a \in A$, $||\pi \circ q(a)|| = ||a||$. In other words, $||a|| = ||a + I||$ for every $a \in A$. Pick a non-zero element $x \in I$. Then the previous equality implies that $||x|| = ||x + I|| = 0$ which is a contradiction. This proves (1) $\implies$ (2).

Suppose (2) holds. Let π be a non-zero representation of A. Then $ker(\pi) = \{0\}$. Hence π is injective. But any injective $*$-homomorphism is isometric. Thus $||\pi(a)|| = ||a||$ for every $a \in A$. Thus (2) $\implies$ (1) is proved. Hence the proof. □

Fix an orthonormal basis $\{\xi_1, \xi_2, \cdots\}$ for $\mathcal{H}$. For i, j, let $E_{ij} = \theta_{\xi_i,\xi_j}$. Observe the following.

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \tag{1.1}$$
$$E_{ij}^* = E_{ji} \tag{1.2}$$

for $i, j, k, l \in \mathbb{N}$. Let $\mathcal{A}_n$ be the linear span of $\{E_{ij} : i, j \in \{1, 2, \cdots, n\}\}$. The above relations imply that $\mathcal{A}_n$ is a $*$-subalgebra of $\mathcal{K}(\mathcal{H})$. Since $\mathcal{A}_n$ is finite dimensional, it follows that $\mathcal{A}_n$ is norm closed. Moreover the map $e_{ij} \to E_{ij}$ from $M_n(\mathbb{C}) \to \mathcal{A}_n$ is an isometric $*$-isomorphism (Why?). Thus $\mathcal{A}_n$ is simple. Observe that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ and $\mathcal{A} := \bigcup_{n \geq 1} \mathcal{A}_n$ is norm dense in $\mathcal{K}(\mathcal{H})$ (Why?).

Proof of Theorem 1.1. Let $\pi : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}_0)$ be a non-zero representation. Since $\mathcal{A}$ is dense in $\mathcal{K}(\mathcal{H})$, it follows that there exists n_0 such that π is non-zero on $\mathcal{A}_{n_0}$. Since $\mathcal{A}_{n_0} \subset \mathcal{A}_n$ for $n \geq n_0$, it follows that π restricted to $\mathcal{A}_n$ is non-zero. But $\mathcal{A}_n$ is simple. Consequently π is isometric on $\mathcal{A}_n$ for $n \geq n_0$. Since $\mathcal{A} = \bigcup_{n \geq n_0} \mathcal{A}_n$, it follows that $||\pi(a)|| = ||a||$ for every $a \in \mathcal{A}$. Since $\mathcal{A}$ is dense in $\mathcal{K}(\mathcal{H})$, it follows that $||\pi(a)|| = ||a||$ for every $a \in \mathcal{K}(\mathcal{H})$. Hence $\mathcal{K}(\mathcal{H})$ is simple. This completes the proof. □

Next we derive a "universal picture" of $\mathcal{K}(\mathcal{H})$. Keep the foregoing notation.

Proposition 1.4 *Let A be a C^*-algebra. Suppose there exists a system of matrix units $\{e_{ij} : i, j \in \mathbb{N}\}$ in A, i.e. the set $\{e_{ij} : i, j \in \mathbb{N}\}$ satisfies the following relations.*

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$
$$e_{ij}^* = e_{ji}$$

for $i, j, k, l \in \mathbb{N}$. Then there exists a unique $*$-homomorphim $\pi : \mathcal{K}(\mathcal{H}) \to A$ such that for $i, j \in \mathbb{N}$, $\pi(E_{ij}) = e_{ij}$.

Proof. Note that $\{E_{ij} : i, j \in \{1, 2, \cdots, n\}\}$ is a basis for $\mathcal{A}_n$ for every n. Thus there exists a linear map $\pi_n : \mathcal{A}_n \to A$ such that $\pi_n(E_{ij}) = e_{ij}$ for $i, j \in \{1, 2, \cdots, n\}$. Clearly π_n is a $*$-homomorphism. Since $\mathcal{A}_n$ is simple, it follows that π_n is isometric. The maps (π_n)'s are consistent, i.e. $\pi_{n+1}|_{\mathcal{A}_n} = \pi_n$. Thus there exists a $*$-homomorphism $\pi : \mathcal{A} \to A$ such that $\pi|_{\mathcal{A}_n} = \pi_n$. Since each π_n is isometric, it follows that π is isometric. Thus, π extends to a $*$-homomorphism to the closure of $\mathcal{A}$ which is $\mathcal{K}(\mathcal{H})$. We denote the extension again by π. It is clear that π is the required map. Uniqueness of π is obvious. □

Derive the following "coordinate free" description of the universal picture of $\mathcal{K}(\mathcal{H})$.

Exercise 1.2 *Let D be a dense subspace of $\mathcal{H}$ and A be a C^*-algebra. Suppose that for $\xi, \eta \in D$, there exists $e_{\xi,\eta} \in A$ such that*

$$e_{\xi,\eta}^* = e_{\eta,\xi}$$
$$e_{\xi_1,\eta_1}e_{\xi_2,\eta_2} = \langle \xi_2 | \eta_1 \rangle e_{\xi_1,\eta_2}$$

for $\xi, \xi_1, \xi_2, \eta, \eta_1, \eta_2 \in D$. Show that there exists a unique $*$-homomorphism $\pi : \mathcal{K}(\mathcal{H}) \to A$ such that $\pi(\theta_{\xi,\eta}) = e_{\xi,\eta}$ for $\xi, \eta \in D$.

Next, we study the representation theory of the algebra of compact operators. The crucial facts regarding the representation theory of compacts are the following:

(1) Any non-degenerate representation of $\mathcal{K}(\mathcal{H})$ is a direct sum of irreducible representations.

(2) The only irreducible representation, up to unitary equivalence, of $\mathcal{K}(\mathcal{H})$ is the identity representation.

This is the content of the next theorem.

Exercise 1.3 *Keep the foregoing notation. Let $E_n = \sum_{i=1}^{n} E_{ii}$. Note that E_n is the projection onto the subspace spanned by $\{\xi_1, \xi_2, \cdots, \xi_n\}$. Hence $E_n \leq E_{n+1}$ for $n \geq 1$. Show the following.*

(1) The sequence $(E_n) \to 1$ strongly, i.e. $E_n\xi \to \xi$ for every $\xi \in \mathcal{H}$.

(2) For every finite rank operator T on $\mathcal{H}$, $TE_n \to T$ and $E_nT \to T$ in norm.

(3) For every compact operator T on $\mathcal{H}$, $TE_n \to T$ and $E_nT \to T$ in norm. In other words, (E_n) is an approximate identity of $\mathcal{K}(\mathcal{H})$.

Lemma 1.5 *The identity representation of $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$ is irreducible.*

Proof. Let W be a non-zero closed subspace of $\mathcal{H}$ which is invariant under $\mathcal{K}(\mathcal{H})$. Pick a unit vector $\eta \in W$. Note that $\theta_{\xi,\eta}(\eta) = \xi$. Thus $\xi \in W$ for every $\xi \in \mathcal{H}$. This implies that $W = \mathcal{H}$. Hence the proof. □

Theorem 1.6 *Let $\pi : \mathcal{K}(\mathcal{H}) \to B(\widetilde{\mathcal{H}})$ be a non-degenerate representation. Then there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to B(\widetilde{\mathcal{H}})$ such that*

$$\pi(A) = U(A \otimes 1)U^*$$

for $A \in \mathcal{K}(\mathcal{H})$.

Proof. Set $E_n := \sum_{i=1}^{n} E_{ii}$. Note that E_n is an approximate identity of $\mathcal{K}(\mathcal{H})$. Since π is non-degenerate, it follows that $\pi(E_n) \to 1$ strongly. Thus there exists i such that $\pi(E_{ii}) \neq 0$. Choose such an i. We claim that $\pi(E_{jj}) \neq 0$ for every j. Note that $\pi(E_{ij})$ is a partial isometry with initial space $\pi(E_{jj})$ and final space $\pi(E_{ii}) \neq 0$. Hence $\pi(E_{jj}) \neq 0$. This proves our claim.

Let $\mathcal{H}_0$ be the range space of $\pi(E_{11})$. Denote the dimension of $\mathcal{H}_0$ by d and let $\{\eta_i\}_{i=1}^d$ be an orthonormal basis for $\mathcal{H}_0$. We claim that $\{\pi(E_{i1})\eta_j\}_{i,j}$ is total in $\widetilde{\mathcal{H}}$. Denote the closed linear span of $\{\pi(E_{i1})\eta_j\}_{i,j}$ by $\mathcal{H}_1$. It is clear that $\pi(E_{rs})$ leaves $\mathcal{H}_1$ invariant for every r, s. Since the linear span of $\{E_{rs}\}$ is norm dense in $\mathcal{K}(\mathcal{H})$, it follows that $\mathcal{H}_1$ is invariant under π and so is $\mathcal{H}_1^\perp$.

Suppose $\mathcal{H}_1^\perp \neq \{0\}$. By definition, it follows that $\mathcal{H}_0 \subset \mathcal{H}_1$. Hence $\mathcal{H}_1^\perp \subset \mathcal{H}_0^\perp = Ker(\pi(E_{11}))$. Thus $\pi(E_{11}) = 0$ on $\mathcal{H}_1^\perp$. But $\pi(E_{i1})$ is a partial isometry with final space $\pi(E_{ii})$ and initial space $\pi(E_{11}) = 0$ on $\mathcal{H}_1^\perp$. Consequently, $\pi(E_{ii}) = 0$ on $\mathcal{H}_1^\perp$ for every i which contradicts the fact that $\pi(E_n) \to 1$ strongly. This proves our claim.

Let $r, s \in \mathbb{N}$ and $j, k \in \{1, 2, \cdots, d\}$ be given. Calculate as follows to observe that

$$\begin{aligned}
\langle \pi(E_{r1})\eta_j | \pi(E_{s1})\eta_k \rangle &= \langle \pi(E_{1s})\pi(E_{r1})\eta_j | \eta_k \rangle \\
&= \delta_{rs} \langle \pi(E_{11})\eta_j | \eta_k \rangle \\
&= \delta_{rs} \langle \eta_j | \eta_k \rangle \\
&= \delta_{rs}\delta_{jk}.
\end{aligned}$$

The above calculation together with the fact that $\{\pi(E_{i1})\eta_j\}_{i,j}$ is total in $\widetilde{\mathcal{H}}$ ensures that there exists a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to \widetilde{\mathcal{H}}$ such that $U(\xi_i \otimes \eta_j) = \pi(E_{i1})\eta_j$. A direct calculation reveals that $U(E_{rs} \otimes 1)U^* = \pi(E_{rs})$. The proof is now completed by appealing to the fact that linear span of $\{E_{ij} : i, j\}$ is dense in $\mathcal{K}(\mathcal{H})$. $\square$

Exercise 1.4 *Let $\pi : \mathcal{K}(\mathcal{H}) \to B(\widetilde{\mathcal{H}})$ be a non-degenerate representation. Suppose there exists a Hilbert space $\mathcal{H}_0$ and a unitary $U : \mathcal{H} \otimes \mathcal{H}_0 \to \widetilde{\mathcal{H}}$ such that*

$$\pi(A) = U(A \otimes 1)U^*$$

for $A \in \mathcal{K}(\mathcal{H})$. Show that $dim(\mathcal{H}_0)$ is the dimension of the range space of $\pi(p)$ where p is any rank one projection. The dimension of $\mathcal{H}_0$ is called the ***the multiplicity*** *of the identity representation in π.*

We usually identify all infinite dimensional separable Hilbert space and reserve the letter $\mathcal{K}$ to indicate the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

2 Universal C^*-algebras

Often, a C^*-algebra is prescribed in terms of generators and relations. We have already seen one example of this phenomenon. The C^*-algebra $\mathcal{K}$ is the universal C^*-algebra generated by a system of "matrix units" $\{e_{ij} : i, j \in \mathbb{N}\}$. We make this idea precise here. This is based on the lectures given by Cuntz in 2014 at Oberwolfach.

Let $\mathcal{A}$ be a $*$-algebra. Let $p : \mathcal{A} \to [0, \infty)$ be a map. We say that p is a C^*-seminorm if

(1) p is seminorm on $\mathcal{A}$,

(2) for $x \in \mathcal{A}$, $p(x^*x) = p(x)^2$, and

(3) for $x, y \in \mathcal{A}$, $p(xy) \leq p(x)p(y)$.

For $x \in \mathcal{A}$, define $||x|| := \sup\{p(x) : \ p \text{ is a } C^*\text{-seminorm on } \mathcal{A}\}$. It is quite possible that $||x||$ is infinite for some $x \in \mathcal{A}$. Suppose assume that $||x|| < \infty$ for every $x \in \mathcal{A}$. Let

$$I := \{x \in \mathcal{A} : ||x|| = 0\}.$$

Condition (3) implies that I is an ideal in $\mathcal{A}$. Consider the quotient $\mathcal{A}/I$. The semi-norm $||\ ||$ descends to a C^*-norm on $\mathcal{A}/I$. The completion of $\mathcal{A}/I$ with respect to this C^*-norm is called the universal C^*-algebra of $\mathcal{A}$ or the enveloping C^*-algebra of $\mathcal{A}$ usually denoted $C^*(\mathcal{A})$.

Exercise 2.1 *Keep the foregoing notation. Show that for every $x \in \mathcal{A}$,*

$$||x|| = \sup\{||\pi(x)|| : \ \pi \textit{ is a } *\textit{-homomorphism from } \mathcal{A} \textit{ to a } C^*\textit{-algebra}\}.$$

$$||x|| = \sup\{||\pi(x)|| : \ \pi \textit{ is a non-degenerate representation of } \mathcal{A}\}.$$

(Recall that a representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is said to be non-degenerate if $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. If $\mathcal{A}$ is unital, non-degenerate representations are precisely unital representations).

Remark 2.1 *Note that $C^*(\mathcal{A})$ exists if and only if $||x|| < \infty$ for every $x \in \mathcal{A}$.*

Consider the natural map $\mathcal{A} \to C^*(\mathcal{A})$. We abuse notation and write the image of an element $x \in \mathcal{A}$ under this map by x itself. The C^*-algebra $C^*(\mathcal{A})$ is called the "universal C^*-algebra of $\mathcal{A}$" because it satisfies the following universal property. Keep the foregoing notation.

Proposition 2.2 *Suppose B is a C^*-algebra and let $\pi : \mathcal{A} \to B$ be a $*$-homomorphism. Then there exists a unique $*$-homomorphism $\widetilde{\pi} : C^*(\mathcal{A}) \to B$ such that $\widetilde{\pi}(x) = \pi(x)$ for every $x \in \mathcal{A}$.*

Proof. Uniqueness is obvious. For existence, let $p : \mathcal{A} \to [0, \infty)$ be defined by $p(x) = ||\pi(x)||$. Note that p is a C^*-seminorm on $\mathcal{A}$. Thus $||\pi(x)|| \leq ||x||$ for every $x \in \mathcal{A}$. This implies that π descends to a $*$-homomorphim say $\widetilde{\pi} : \mathcal{A}/I \to B$. It is clear that $\widetilde{\pi}$ is bounded. Denote the extension to $C^*(\mathcal{A})$ again by $\widetilde{\pi}$. Then $\widetilde{\pi}$ is the required map. This completes the proof. □

Often, the algebra $\mathcal{A}$ itself is given by generators and relations. For example, consider the following statements

(1) Let $\mathcal{A}$ be the universal unital $*$-algebra generated by a single element u such that $u^*u = 1$ and $uu^* = 1$.

(2) Let $\mathcal{A}$ be the universal unital $*$-algebra generated by v such that $v^*v = 1$.

(3) Let $\mathcal{A}$ be the universal unital $*$-algebra generated by P, Q such that $PQ - QP = 1$.

(4) Let $\mathcal{A}$ be the universal $*$-algebra generated by $\{p_i\}_{i=1}^n$ such that $p_i^2 = p_i = p_i^*$ and $p_ip_j = \delta_{ij}p_i$.

What do we mean in each statement ? For example in (1), we mean that there exists a $*$-algebra $\mathcal{A}$, unique up to unique isomorphism, which is generated by a single element u and has the following universal property : Suppose $\mathcal{B}$ is a unital $*$-algebra and $w \in \mathcal{B}$ be such that $w^*w = ww^* = 1$. Then there exists a unique $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi(u) = w$. We do the same for (2), (3) and (4). The justification of the existence of such an $\mathcal{A}$ is always by abstract nonsense.

Consider now the following statement. Let $\mathcal{T}$ be the universal unital C^*-algebra generated by a single element v such that $v^*v = 1$. What do we mean by this ? First, we take the universal unital $*$-algebra generated by v such that $v^*v = 1$. Denote it by $\mathcal{A}$. Then $\mathcal{T} = C^*(\mathcal{A})$. The C^*-algebra $\mathcal{T}$ is called the *Toeplitz algebra* in the literature. But, does $\mathcal{T}$ exist ? Yes, it exists. For suppose p is a C^*-seminorm on $\mathcal{A}$. Define $I_p := \{a \in \mathcal{A} : p(a) = 0\}$. Then p descends to a C^*-norm on $\mathcal{A}/I_p$. Let A_p be the completion of $\mathcal{A}/I_p$. Since $v + I_p$ is an isometry in A_p and consequently $||v + I_p|| \leq 1$. Thus any word in v and v^* has p-norm at most 1. Now let x be an element in $\mathcal{A}$. Write $x = \sum_{x_\alpha} w_\alpha$ where w_α is a word in v and v^*. Then $p(x) \leq \sum_\alpha |x_\alpha|$ and the latter bound is independent of p. Consequently $||x|| < \infty$ for every x.

Remark 2.3 *The argument outlined above works in the following situtation. Suppose $\mathcal{A}$ is a $*$-algebra generated by $\{x_i\}$ and each x_i has p-norm atmost 1 for every C^*-seminorm p on $\mathcal{A}$. Then $C^*(\mathcal{A})$ exists. However $C^*(\mathcal{A})$ might be zero.*

The Toeplitz algebra has the following "universal property"

Exercise 2.2 *Suppose B is a unital C^*-algebra and $w \in B$ is such that $w^*w = 1$. Then there exists a unique $*$-homomorphims $\pi : \mathcal{T} \to B$ such that $\pi(v) = w$.*

Let us now show that the Toeplitz algebra is non-zero. We show this by producing a non-zero representation of $\mathcal{T}$. Consider the Hilbert space $\ell^2(\mathbb{N})$. Let $\{\delta_n\}_{n\geq 1}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the unique operator such that $S(\delta_n) = \delta_{n+1}$. Then S is an isometry, i.e. $S^*S = 1$. The universal property of $\mathcal{T}$ guarantees that there exists a unique $*$-homomorphism $\pi : \mathcal{T} \to C^*(S)$ such that $\pi(v) = S$. Later, we will show that π is an isomorphism.

To summarise, we usually, but not always, apply Remark 2.3 to justify the existence of the universal C^*-algebra. To show, it is non-zero, we need to find a non-zero representation of the universal $*$-algebra $\mathcal{A}$ on a Hilbert space or equivalently a non-zero $*$-homomorphism from $\mathcal{A}$ to a C^*-algebra. Use this to do the following exercises.

Exercise 2.3 *Show that the universal unital C^*-algebra generated by u such that $u^*u = uu^* = 1$ exists.*

Exercise 2.4 *Show that the universal unital C^*-algebra generated by $\{p_i\}_{i=1}^n$ satisfying the relations $p_i^2 = p_i = p_i^*$ and $p_ip_j = \delta_{ij}p_i$ exists.*

Let us identify the universal C^*-algebras considered in the above two exercises concretely.

Proposition 2.4 *The algebra of continuous functions on the circle $\mathbb{T}$ denoted $C(\mathbb{T})$ is the universal C^*-algebra generated by u such that $u^*u = uu^* = 1$.*

Proof. We denote the function $\mathbb{T} \ni z \to z \in \mathbb{C}$ by z itself. Let A be the universal C^*-algebra generated by u such that $u^*u = uu^* = 1$. Note that u is a unitary in A. The continuous functional calculus gives a $*$-homomorphism $C(\mathbb{T}) \to A$ which maps $z \to u$. Call it ρ. The universal property of A gives a map $\pi : A \to C(\mathbb{T})$ such that $\pi(u) = z$. It is clear that $\pi \circ \rho(z) = u$ and $\rho \circ \pi(u) = z$. Since u and z generates A and $C(\mathbb{T})$ respectively, it follows that π and ρ are inverses of each other. This completes the proof. □

Proposition 2.5 *Let A be the universal C^*-algebra generated by $\{p_i : i \in \mathbb{N}\}$ such that $p_i^2 = p_i = p_i^*$ and $p_ip_j = \delta_{ij}p_i$. Then $A \simeq C_0(\mathbb{N})$.*

Proof. We leave the proof that A exists to the reader. Let $e_i \in C_0(\mathbb{N})$ be such that the ith coordinate of e_i is 1 and the rest of the coordinates are zero. It is clear that $e_i^2 = e_i = e_i^*$ and $e_ie_j = \delta_{ij}e_i$. By the universal property, there exists a $*$-homomorphism $\pi : A \to C_0(\mathbb{N})$ such that $\pi(p_i) = e_i$.

Claim: Let B be a C^*-algebra and $q_1, q_2, \cdots, q_n$ be a finite sequence of orthogonal projections. Then for every $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$,

$$||\sum_{i=1}^{n} \lambda_i q_i|| \leq \sup_{1 \leq i \leq n} |\lambda_i|.$$

By representing B faithfully on a Hilbert space say $\mathcal{H}$, we can assume that $q_1, q_2, \cdots, q_n$ are operators on $\mathcal{H}$. Then for a unit vector $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
||\sum_{i=1}^{n} \lambda_i q_i \xi||^2 &= \sum_{i=1}^{n} |\lambda_i|^2 \langle q_i \xi | \xi \rangle \\
&\leq (\sup_{1 \leq i \leq n} |\lambda_i|)^2 (\langle \sum_{i=1}^{n} q_i \xi | \xi \rangle \\
&\leq (\sup_{1 \leq i \leq n} |\lambda_i|)^2 \text{ (since } \sum_{i=1}^{n} q_i \text{ is a projection).}
\end{aligned}$$

Hence

$$||\sum_{i=1}^{n} \lambda_i q_i|| \leq \sup_{1 \leq i \leq n} |\lambda_i|. \tag{2.3}$$

Consider the dense $*$-subalgebra $C_c(\mathbb{N})$ of $C_0(\mathbb{N})$. Note that $\{e_i : i \in \mathbb{N}\}$ is a basis for $C_c(\mathbb{N})$. Let $\rho : C_c(\mathbb{N}) \to A$ be the linear map such that $\rho(e_i) = p_i$. Clearly ρ is a $*$-homomorphism. The estimate 2.3 implies that ρ is bounded. Denote the extension of ρ to $C_0(\mathbb{N})$ by ρ itself. Then $\rho \circ \pi$ agrees with the identity map on the generators. Consequently $\rho \circ \pi$ is identity. Similarly $\pi \circ \rho$ is identity. This shows that ρ and π are inverses of each other. Hence π is an isomorphism. This completes the proof. □

Let us give a non-example. The universal C^*-algebra generated by two elements P, Q such that $PQ - QP = 1$ is zero. It suffices to show the following.

Proposition 2.6 *Let $\mathcal{H}$ be a non-zero Hilbert space. Then there does not exist bounded operators P and Q on $\mathcal{H}$ such that $PQ - QP = 1$.*

Proof. Suppose, on the contrary, assume that there exist $P, Q \in B(\mathcal{H})$ such that the commutator $[P,Q] = PQ - QP = 1$. For a bounded operator T, let $\sigma(T)$ be the spectrum of T. Recall that for bounded operators T, S, $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

Choose $\lambda \in \sigma(QP)$. Note that $\lambda + 1 \in \sigma(QP + 1) = \sigma(PQ) \subset \sigma(PQ) \cup \{0\} \subset \sigma(QP) \cup \{0\}$. Suppose $\lambda \in \{-1, -2, -3, \cdots\}$. Then $\lambda + k \in \sigma(QP) \cup \{0\}$ for every positive integer k. The compactness of $\sigma(QP)$ implies that there exists a positive integer k such that $\lambda = -k$. The fact that $\lambda \in \sigma(QP) \implies \lambda + 1 \in \sigma(QP) \cup \{0\}$ implies that $-1 \in \sigma(QP)$.

Then $-1 \in \sigma(PQ)$. The relation $PQ - QP = 1$ implies that $-2 \in \sigma(QP)$ which in turn implies $-2 \in \sigma(PQ)$. By induction, we obtain $-k \in \sigma(PQ)$ for every positive integer k which contradicts the fact that $\sigma(PQ)$ is bounded. This completes the proof. □

When one talks of the universal C^*-algebra given in terms of generators and relations, one should be cautious and decide first whether it exists or not and whether it is zero or is non-zero. The notion of universal C^*-algebra is very handy and allows us to quickly define group C^*-algebras and crossed products of discrete groups which provide important examples of C^*-algebras.

Group C^*-algebras: Let G be a discrete group.[1] Then $C^*(G)$, called *the full group C^*-algebra of G*, is defined to be the universal C^*-algebra generated by $\{u_s : s \in G\}$ which satisfy the following relations:

$$\begin{aligned} u_s u_t &= u_{st} \\ u_s^* &= u_{s^{-1}} \end{aligned}$$

for $s, t \in G$. Note that the above relations imply that u_e is a multiplicative identity of $C^*(G)$ where e is the identity element of G. Moreover $\{u_s : s \in G\}$ is a family of "unitaries" and consequently $||u_s|| \leq 1$ for every $g \in G$. Thus, by Remark 2.3, it follows that $C^*(G)$ exists. The next thing to show is that $C^*(G)$ is non-zero.

Consider the Hilbert space $\ell^2(G)$ and let $\{\epsilon_t : h \in G\}$ be an orthonormal basis for $\ell^2(G)$. For $s \in G$, let λ_s be the unitary operator on $\ell^2(G)$ such that

$$\lambda_s(\epsilon_t) = \epsilon_{st}$$

for $t \in G$. The map $G \ni s \to \lambda_s \in B(\ell^2(G))$ is called *the left regular representation of G*. Then clearly $\lambda_s \lambda_t = \lambda_{st}$ and $\lambda_s^* = \lambda_{s^{-1}}$. Thus there exists a unique unital $*$-

[1] We always assume some sort of separability hypothesis. For instance, we mostly assume topological spaces are second countable, discrete groups are second countable, Hilbert spaces are separable etc... We do things as if this hypothesis is always there and make no explicit mention of this.

homomorphism $\widetilde{\lambda} : C^*(G) \to B(\ell^2(G))$ such that $\widetilde{\lambda}(u_s) = \lambda_s$. This shows that $C^*(G)$ is non-zero. The image of π is a C^*-subalgebra of $B(\ell^2(G))$ and is called *the reduced C^*-algebra of G* and is denoted $C^*_{red}(G)$. Note that $C^*_{red}(G)$ is the C^*-algebra generated by $\{\lambda_s : s \in G\}$. Sometimes, we abuse notation and write $\widetilde{\lambda}$ simply by λ. It is natural to ask whether $\widetilde{\lambda}$ is an isomorphism. It turns out that $\widetilde{\lambda}$ is an isomorphism if and only if the group G is *amenable*. Abelian groups are amenable. An example of a non-amenable group is the free group on 2 generators $\mathbb{F}_2$.

Let us take a closer look at $C^*(G)$. Let $\mathcal{A}$ be the universal $*$-algebra generated by $\{u_s\}_{s \in G}$ such that $u_s u_t = u_{st}$ and $u_s^* = u_{s^{-1}}$. We first obtain a concrete description of $\mathcal{A}$. Let $C_c(G)$ denote the space of finitely supported complex valued functions on G. Define a $*$-algebraic structure on $C_c(G)$ as follows. For $f, g \in C_c(G)$, let $f * g : G \to \mathbb{C}$ be defined by

$$f * g(s) = \sum_{t \in G} f(st) g(t^{-1}).$$

Note that $f*g$ is well defined. For f and g are finitely supported. Also $f*g \in C_c(G)$. The multiplication operation defined above is called *the convolution*. Define a $*$-operation on $C_c(G)$ by $f^*(s) = \overline{f(s^{-1})}$.

Exercise 2.5 *Show that $C_c(G)$ with the convolution and the $*$-operation defined above is a $*$-algebra.*

The algebra $C_c(G)$ is usually called *the group algebra of G* and the usual notation is $C[G]$. For $s \in G$, let $\delta_s \in C_c(G)$ be given by

$$\delta_s(t) := \begin{cases} 1 & \text{if } t = s \in X, \\ \\ 0 & \text{if } t \neq 0. \end{cases} \tag{2.4}$$

Observe that $\delta_s * \delta_t = \delta_{st}$ and $\delta_s^* = \delta_{s^{-1}}$. Note that δ_e is the multiplicative identity of $C_c(G)$. Thus, there exists a $*$-homomorphism $\pi : \mathcal{A} \to C_c(G)$ such that $\pi(u_s) = \delta_s$ for $s \in G$.

Lemma 2.7 *The map π is an isomorphism.*

Proof. We define the inverse map directly by setting $\rho(\delta_s) = u_s$. This is possible provided we can show that $\{\delta_s : s \in G\}$ is a basis for $C_c(G)$. We claim that $\{\delta_s : s \in G\}$ is a basis for $C_c(G)$. Let $f \in C_c(G)$ be given then $f = \sum_{s \in G} f(s)\delta_s$. Moreover if $f = \sum_{s \in G} a_s \delta_s$, then applying the equality at an arbitrary point t, we get $f(t) = a_t$. This proves our claim. Let $\rho : C_c(G) \to \mathcal{A}$ be the linear map such that $\rho(\delta_s) = u_s$. Then clearly $\rho \circ \pi$ and $\pi \circ \rho$ agrees with identity maps on the generators and hence agrees with the identity maps everywhere. This shows that ρ and π are inverses of each other. Hence the proof. $\square$

Thus $C^*(G)$ is the completion of $C_c(G)$ where the norm on $C_c(G)$ is given by

$$||f|| := \sup\{\pi(f) : \ \pi \text{ is a unital representation of } C_c(G) \text{ on a Hilbert space}\}.$$

How does representations of $C_c(G)$ arise ? Let $\mathcal{H}$ be a Hilbert space and $U : G \to B(\mathcal{H})$ be a map. We say that U is a unitary representation if

(1) for $s, t \in G$, $U_s U_t = U_{st}$, and

(2) for $s \in G$, U_s is a unitary.

The set of unitaries $\mathcal{U}(\mathcal{H})$ is a group and a unitary representation of G on $\mathcal{H}$ is simply a group homomorphism from G to $\mathcal{U}(\mathcal{H})$. Let $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then $U_s^* = U_{s^{-1}}$ for $s \in G$. Thus there exists a unique unital $*$-homomorphism, denoted $\pi_U : C_c(G) \to B(\mathcal{H})$ such that

$$\pi_U(\delta_s) = U_s$$

for $s \in G$. Conversely, suppose π is a unital representation of $C_c(G)$ on a Hilbert space $\mathcal{H}$. Set $U_s = \pi(\delta_s)$. Then $\{U_s\}_{s \in G}$ is a unitary representation of G. Clearly π_U and π agrees on $\{\delta_s : s \in G\}$. Since $\{\delta_s : s \in G\}$ is a basis for $C_c(G)$, it follows that $\pi = \pi_U$. Thus, representations of $C_c(G)$ are the "same" as the unitary representations of the group G. Thus for $f \in C_c(G)$,

$$||f||_{C^*(G)} := \sup\{\pi_U(f) : U \text{ is a unitary representation of } G\}.$$

Representations of $C^*(G)$ are in one-one correspondence with representations of $C_c(G)$. To summarise, $C^*(G)$ is the C^*-algebra that captures the representation theory of the group G.

Remark 2.8 *The map $U \to \pi_U$ respects unitary equivalence, irreducibility, direct sum, etc..... Thus, the study of the representation theory of groups is equivalent to the study of the representation theory of the associated full group C^*-algebra. This has advantages, for then we can use (operator) algebraic techniques. My favourite application of this philosophy is the proof of the fact that a finite group admits only finitely many irreducible representations, up to unitary equivalence.*

This is because if G is a finite group then $C^*(G)$ is a finite dimensional C^*-algebra. Consequently, $C^*(G)$ is a direct sum of "matrix algebras". But, for every n, $M_n(\mathbb{C})$ has only one irreducible representation up to equivalence.

Let us identify the full C^*-algebra of a discrete abelian group. Let G be a discrete abelian group. Denote the set of homomorphisms from G to the multiplicative group $\mathbb{T}$ by $\widehat{G}$. The set $\widehat{G}$ has a group structure where the group multiplication is pointwise multiplication. The map $G \ni s \to 1 \in \mathbb{T}$ is the identity element of G. For $\chi \in \widehat{G}$, the inverse of χ is $\overline{\chi}$. We endow $\widehat{G}$ with the topology of pointwise convergence, i.e. the product topology. The convergence of nets is as follows. Suppose (χ_α) is a net in $\widehat{G}$ and $\chi \in \widehat{G}$. Then $\chi_\alpha \to \chi$ if and only if $\chi_\alpha(s) \to \chi(s)$ for every $s \in G$. By Tychonoff theorem, it follows that $\widehat{G}$ is compact. It is routine to check that $\widehat{G}$ is a topological group.

Proposition 2.9 *Let G be a discrete abelian group. Then $C^*(G)$ is isomorphic to $C(\widehat{G})$.*

Proof. Since the group G is abelian, it follows that $A := C^*(G)$ is commutative. Moreover A is unital. It suffices to show that $\widehat{A}$ is homeomorphic to $\widehat{G}$. Let $\chi \in \widehat{G}$ be given. Then, by the universal property, there exists a homomorphism $\overline{\chi} : A \to \mathbb{C}$ such that $\overline{\chi}(u_s) = \chi(s)$. Since $\{u_s : s \in G\}$ generates $C^*(G)$, the map

$$\widehat{G} \ni \chi \to \overline{\chi} \in \widehat{A}$$

is 1-1. Let $\omega : A \to \mathbb{C}$ be a character. Since $\{u_s : s \in G\}$ is a set of unitaries, it follows that for every $s \in G$, $\omega(u_s) \in \mathbb{T}$. Set $\chi : G \to \mathbb{T}$ by $\chi(s) = \omega(u_s)$. It is clear that χ is a character of G and $\omega = \overline{\chi}$. This proves that the map $\widehat{G} \ni \chi \to \overline{\chi} \in \widehat{A}$ is onto.

Let (χ_α) be a net in $\widehat{G}$ such that $(\chi_\alpha) \to \chi \in \widehat{G}$. Note that $\{\overline{\chi_\alpha}\}$ is uniformly bounded. Thus to show $\overline{\chi_\alpha} \to \overline{\chi}$, it suffices to check $\overline{\chi_\alpha}(x) \to \overline{\chi}(x)$ for x in a total set F of A. Set $F := \{u_s : s \in G\}$ and observe that F is total in A. Clearly $\overline{\chi_\alpha}(x) \to \overline{\chi}(x)$ for every $x \in F$. Hence the map $\widehat{G} \ni \chi \to \overline{\chi} \in \widehat{A}$ is continuous. Since $\widehat{G}$ and $\widehat{A}$ are both compact Hausdorff, it follows that the map $\chi \to \overline{\chi}$ is a homeomorphism. This completes the proof. $\square$

Crossed products: Let G be a discrete group and let A be a C^*-algebra. By an *action of G on A*, we mean a family $\alpha := \{\alpha_s\}_{s \in G}$ of automorphisms of A such that $\alpha_s \circ \alpha_t = \alpha_{st}$ for $s, t \in G$. Such a triple (A, G, α) is called a *C^*-dynamical system.* Here is an example of a dynamical system.

Example 2.10 *Let X be a locally compact Hausdorff space and G be a discrete group which acts on X via homeomorphisms on the left. For $s \in G$ and $f \in C_0(X)$, define*

$$\alpha_s(f)(x) = f(s^{-1}x).$$

Then $\alpha := \{\alpha_s\}_{s\in G}$ is an action of G on $C_0(X)$. Note that G leaves $C_c(X)$ invariant where $C_c(X)$ denotes the dense subalgebra of compactly supported continuous functions on X.

Let G be a discrete group and let A be a unital C^*-algebra. Suppose G acts on A and let α be the action. The *full crossed product*, denoted $A \rtimes_\alpha G$, is defined to be the universal unital C^*-algebra generated by a copy of A and unitaries $\{u_s\}_{s\in G}$ such that $u_s u_t = u_{st}$ and $u_s a u_s^* = \alpha_s(a)$ for $s \in G$ and $a \in A$. Note that if $A = \mathbb{C}$ and α is the trivial action then $A \rtimes_\alpha G \simeq C^*(G)$.

Let us take a closer look at the C^*-algebra $A \rtimes_\alpha G$. First, let $\mathcal{B}$ be the universal $*$-algebra generated by a copy of A and unitaries $\{u_s\}_{s\in G}$. The relations imply that the linear span of $\{a_s u_s : a_s \in A, s \in G\}$ is $\mathcal{B}$. Our experience with group C^*-algebras suggest that we should treat $\mathcal{B}$ as the algebra of functions defined on G but now taking values in A. Thus consider $C_c(G, A)$, i.e. the set of functions $f : G \to A$ such that f is finitely supported.

We make $C_c(G, A)$ into a $*$-algebra by defining the multiplication and the $*$-operation as follows: for $f, g \in C_c(G, A)$,

$$f * g(s) = \sum_{t\in G} f(t)\alpha_t(g(t^{-1}s))$$
$$f^*(s) = \alpha_s(f(s^{-1})^*)$$

for $f, g \in C_c(G, A)$. It is a tedious but a routine exercise to verify that the multiplication and the $*$-operation defined above makes $C_c(G, A)$ into a $*$-algebra. Note that for $f, g \in G$, $f * g(s) = \sum_{t\in G} f(st^{-1})\alpha_{st^{-1}}(g(t))$. For $a \in A$ and $s \in G$, denote the element of $C_c(G, A)$ which vanishes at points other than s and whose value at s is a by $a \otimes \delta_s$. Note that for $f \in C_c(G, A)$, $f = \sum_{s\in G} f(s) \otimes \delta_s$.

Exercise 2.6 *Keep the foregoing notation. Prove that for $a, b \in A$ and $s, t \in G$,*

$$(a \otimes \delta_s) * (b \otimes \delta_t) = a\alpha_s(b) \otimes \delta_{st}$$
$$(a \otimes \delta_s)^* = \alpha_{s^{-1}}(a^*) \otimes \delta_{s^{-1}}$$

The above relations and the universal property of $\mathcal{B}$ together imply that there exists a $*$-homomorphism $\lambda : \mathcal{B} \to C_c(G, A)$ such that $\lambda(a) = a \otimes \delta_e$ and $\lambda(u_s) = 1 \otimes \delta_s$. Let $\mu : C_c(G, A) \to \mathcal{B}$ be defined by $\mu(f) = \sum_{s \in G} f(s)u_s$. Note that μ is a $*$-homomorphism. (The multiplication and the $*$-operation are defined in such a way on $C_c(G, A)$ precisely to make this map a homomorphism). Clearly $\lambda \circ \mu = Id$. Note that $\mu \circ \lambda$ agrees with the identity map on A and $\{u_s : s \in G\}$ which generates $\mathcal{B}$ as an algebra. Thus $\mu \circ \lambda = Id$. Hence λ and μ are inverses of each other.

Then $A \rtimes G$ is the enveloping C^*-algebra of $C_c(G, A)$. Note that the $*$-algebra $C_c(G, A)$ makes sense even if A is not unital.

Definition 2.11 *Suppose G is a discrete group, A is a C^*-algebra and $\alpha := \{\alpha_s\}_{s \in G}$ is an action of G on A. The full crossed product, denoted $A \rtimes_\alpha G$, is defined as the enveloping C^*-algebra of $C_c(G, A)$.*

We need to show that $A \rtimes_\alpha G$ exists and is non-zero. This requires us to prove that the universal norm is finite and we are forced to understand non-degenerate representations of $C_c(G, A)$ in more concrete terms. We will make use of the following remark in the sequel.

Remark 2.12 *We will repeatedly make use of the following. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and S_1 and S_2 are total subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $\phi : S_1 \to S_2$ be a map such that $\langle \phi(x)|\phi(y)\rangle = \langle x|y\rangle$ for $x, y \in S_1$. Then there exists a unique isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ which extends ϕ. Moreover if ϕ is a bijection, the isometry V is a unitary.*

Let $\lambda : C_c(G, A) \to B(\mathcal{H})$ be a non-degenerate representation. Since $\{a \otimes \delta_s : a \in A, s \in G\}$ spans $C_c(G, A)$, it follows that $\{\lambda(a \otimes \delta_s)\xi : a \in A, s \in G, \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$. Fix $r \in G$. For $a, b \in A$, $s, t \in G$ and $\xi, \eta \in \mathcal{H}$, calculate as follows to observe that

$$\begin{aligned}
&\langle \lambda(\alpha_r(a) \otimes \delta_{rs})\xi | \lambda(\alpha_r(b) \otimes \delta_{rt})\eta\rangle \\
&= \langle \xi | \lambda(\alpha_r(a) \otimes \delta_{rs})^* \lambda(\alpha_r(b) \otimes \delta_{rt})\eta\rangle \\
&= \langle \xi | \lambda((\alpha_{s^{-1}r^{-1}}(\alpha_r(a^*)) \otimes \delta_{s^{-1}r^{-1}}) * (\alpha_r(b) \otimes \delta_{rt}))\eta\rangle \\
&= \langle \xi | \lambda(\alpha_{s^{-1}}(a^*b) \otimes \delta_{s^{-1}t})\eta\rangle \\
&= \langle \xi | \lambda(a \otimes \delta_s)^* \lambda(b \otimes \delta_t)\eta\rangle \\
&= \langle \lambda(a \otimes \delta_s)\xi | \lambda(b \otimes \delta_t)\eta\rangle.
\end{aligned}$$

Appealing to Remark 2.12, we conclude that there exists a unique unitary, denoted U_r, such that $U_r(\lambda(a \otimes \delta_s)\xi) = \lambda(\alpha_r(a) \otimes \delta_{rs})\xi$ for $a \in A$, $s \in G$ and $\xi \in \mathcal{H}$. By evaluating on the total set $\{\lambda(a \otimes \delta_s)\xi : a \in A, s \in G, \xi \in \mathcal{H}\}$, we conclude that $U_r U_s = U_{rs}$ for every $r, s \in G$. Thus $U := \{U_s\}_{s \in G}$ is a unitary representation of G on $\mathcal{H}$.

Define for $a \in A$, $\pi(a) = \lambda(a \otimes \delta_e)$. Then π is a $*$-representation of A on $\mathcal{H}$.

Exercise 2.7 *Prove by evaluating on the total set* $\{\lambda(a \otimes \delta_s)\xi : a \in A, s \in G, \xi \in \mathcal{H}\}$ *that*

(1) the representation π is a $*$-representation,

(2) the family $U := \{U_s\}_{s \in G}$ is a unitary representation of G,

(3) for $a \in A$ and $s \in G$, $U_s \pi(a) U_s^* = \pi(\alpha_s(a))$ or equivalently $U_s \pi(a) = \pi(\alpha_s(a)) U_s$.

Such a pair (π, U) is called a covariant representation of the dynamical system.

Keep the foregoing notation. The representation λ can be recovered from the pair (π, U). Note, again by evaluating on the total set, that $\lambda(a \otimes \delta_s) = \pi(a)U_s$. Since the set $\{\lambda(a \otimes \delta_s)\xi : a \in A, s \in G, \xi \in \mathcal{H}\}$ is total, it follows that $\{\pi(a)U_s\xi : a \in A, s \in G, \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$. This implies that the representation π is non-degenerate.

We can reverse the above process. First a definition.

Definition 2.13 *Consider a C^*-dynamical system (A, G, α). Let $\pi : A \to B(\mathcal{H})$ be a representation and $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. We say that the pair (π, U) is a covariant representation of the dynamical system (A, G, α) if for $a \in A$, $s \in G$,*

$$U_s \pi(a) U_s^* = \pi(\alpha_s(a)).$$

We always assume that π is non-degenerate.

Let (π, U) be a covariant representation of the dynamical system (A, G, α). Define a map $\lambda : C_c(G, A) \to B(\mathcal{H})$ by $\lambda(f) = \sum_{s \in G} \pi(f(s))U_s$. It is clear that $\lambda(f * g) = \lambda(f)\lambda(g)$ and $\lambda(f)^* = \lambda(f^*)$ if f and g are of the form $a \otimes \delta_s$. But since $\{a \otimes \delta_s\}$ spans $C_c(G, A)$, it follows that λ is a $*$-homomorphism. Note that $\lambda(a \otimes \delta_e) = \pi(a)$. Hence λ is non-degenerate. We denote this map λ by $\pi \rtimes U$.

Exercise 2.8 *Let (π, U) be a covariant representation of (A, G, α). Prove that the covariant representation that we obtain if we apply the process described before Definition 2.13 to the non-degenerate representation $\pi \rtimes U$ is (π, U).*

Thus non-degenerate representations of the $*$-algebra $C_c(G,A)$ are in 1-1 correspondence with covariant representations of the dynamical system (A,G,α). Therefore, the universal norm on $C_c(G,A)$ is given by

$$||f|| = \sup\{||(\pi \rtimes U)(f) : (\pi,U) \text{ is a covariant representation of } (A,G,\alpha)\}.$$

For $f \in C_c(G,A)$, let $||f||_1 := \sum_{s\in G}||f(s)||$. Let (π,U) be a covariant representation of (A,G,α). Note that for $f \in C_c(G,A)$,

$$||(\pi \rtimes U)(f)|| = ||\sum_{s\in G}\pi(f(s))U_s|| \leq \sum_{s\in G}||\pi(f(s))||||U_s|| \leq \sum_{s\in G}||f(s)|| = ||f||_1.$$

Hence $||f|| \leq ||f||_1$ for every $f \in C_c(G,A)$. This proves that $||\ ||$ is a genuine C^*-seminorm on $C_c(G,A)$. Next we show that $||\ ||$ is indeed a norm on $C_c(G,A)$ by exhibiting a covariant representation.

Let $\pi : A \to B(\mathcal{H})$ be a faithful representation. Consider the Hilbert space $\widetilde{\mathcal{H}} := \mathcal{H}\otimes\ell^2(G)$. Let $\{\epsilon_t : t \in G\}$ be the standard orthonormal basis for $\ell^2(G)$. For $a \in A$, let $\widetilde{\pi}(a)$ be the bounded operator on $\widetilde{\mathcal{H}}$ given by the equation $\widetilde{\pi}(a)(\xi\otimes\epsilon_t) = \pi(\alpha_t^{-1}(a))\otimes\epsilon_t$. Let $\{\lambda_s : s \in G\}$ be the left regular representation of $\ell^2(G)$. For $s \in G$, set $\widetilde{\lambda}_s = 1\otimes\lambda_s$.

Exercise 2.9 *Verify that $(\widetilde{\pi},\widetilde{\lambda})$ is a covariant representation of (A,G,α).*

Proposition 2.14 *The map $C_c(G,A) \ni f \to (\widetilde{\pi}\rtimes\widetilde{\lambda})(f) \in B(\widetilde{\mathcal{H}})$ is injective.*

Proof. Suppose $(\widetilde{\pi}\rtimes\widetilde{\lambda})(f) = 0$. For $s \in G$, set $a_s = f(s)$. Then for every $\xi,\eta \in \mathcal{H}$ and $r,t \in G$, we have $\sum_{s\in G}\langle\widetilde{\pi}(a_s)\widetilde{\lambda}_s(\xi\otimes\epsilon_r)|\eta\otimes\epsilon_t\rangle = 0$. This implies that for $\xi,\eta \in \mathcal{H}$ and $r,t \in G$,

$$\sum_{s\in G}\langle\pi(\alpha_{sr}^{-1}(a_s))\xi\otimes\epsilon_{sr}|\eta\otimes\epsilon_t\rangle = 0.$$

Fix $s \in G$. In the previous expansion, substitute $r = s^{-1}$ and $t = e$ to obtain $\langle\pi(a_s)\xi|\eta\rangle = 0$ for every $\xi,\eta \in \mathcal{H}$. But π is faithful. This implies that $a_s = 0$. Hence $f = 0$. This completes the proof. $\square$

Keep the foregoing notation. For $f \in C_c(G,A)$, define $||f||_{red} = ||(\widetilde{\pi}\rtimes\widetilde{\lambda})(f)||$. By what we have shown, it follows that $||\ ||_{red}$ is a C^*-norm on $C_c(G,A)$. By definition, $||f||_{red} \leq ||f||$ for $f \in C_c(G,A)$. Hence the universal norm $||\ ||$ is a C^*-norm.

Definition 2.15 *The completion of $C_c(G,A)$ with respect to the universal norm $||\ ||$ is called the full crossed product and is denoted $A \rtimes_\alpha G$.*

Remark 2.16 *It is a remarkable fact that* $|| \ ||_{red}$ *is independent of the chosen faithful representation* π. *We will prove this in the next chapter. The norm* $|| \ ||_{red}$ *is called the reduced norm on* $C_c(G, A)$. *The completion of* $C_c(G, A)$ *with respect to the reduced norm is called the reduced crossed product and is denoted* $A \rtimes_{r,\alpha} G$.

Clearly there is a natural surjective homomorphism from $A \rtimes_\alpha G \to A \rtimes_{r,\alpha} G$. *Unless there is some amenability hypothesis, we cannot expect the above map to be an isomorphism.*

Exercise 2.10 *Suppose* $\mathcal{A}$ *is a dense* $*$*-algebra of* A *and assume that* $\alpha_s(\mathcal{A}) \subset \mathcal{A}$ *for every* $s \in G$. *Prove that* $\mathcal{A} \rtimes_\alpha G := span\{a \otimes \delta_s : a \in \mathcal{A}, s \in G\}$ *is a dense* $*$*-subalgebra of* $A \rtimes_\alpha G$.

Let us identity one example of a crossed product explicity. Let G be a discrete group and let G acts on the topological space G by left translations. Consider the induced action α of G on $C_0(G)$. For $s \in G$, let χ_s be the characteristic function at s. Then $\chi_s \in C_c(G)$ and $\alpha_s(\chi_t) = \chi_{st}$.

Proposition 2.17 *The crossed product* $C_0(G) \rtimes_\alpha G$ *is isomorphic to* $\mathcal{K}(\ell^2(G))$.

Proof. The algebra $\mathcal{K}(\ell^2(G))$ has a universal picture. Thus, it suffices to exhibit appropriate matrix units in $C_0(G) \rtimes_\alpha G$. Let $\{E_{s,t} : s, t \in G\}$ be the natural system of matrix units in $\mathcal{K}(\ell^2(G))$ which correspond to the standard orthonormal basis $\{\epsilon_t : t \in G\}$ of $\ell^2(G)$.

For $s, t \in G$, let $e_{s,t} := \chi_s \otimes \delta_{st^{-1}}$. For $q, r, s, t \in G$, calculate as follows to observe that

$$\begin{aligned} e_{q,r} * e_{s,t} &= (\chi_q \otimes \delta_{qr^{-1}}) * (\chi_s \otimes \delta_{st^{-1}}) \\ &= \chi_q \chi_{qr^{-1}s} \otimes \delta_{qr^{-1}st^{-1}} \\ &= \delta_{q,qr^{-1}s} \chi_q \otimes \delta_{qt^{-1}} \\ &= \delta_{r,s} \chi_q \otimes \delta_{qt^{-1}} \\ &= \delta_{r,s} e_{q,t}. \end{aligned}$$

For $s, t \in G$, observe that

$$e_{s,t} = (\chi_s \otimes \delta_{st^{-1}})^* = \alpha_{ts^{-1}}(\chi_s) \otimes ts^{-1} = \chi_t \otimes \delta_{ts^{-1}} = e_{t,s}.$$

Thus $\{e_{s,t} : s, t \in G\}$ forms a system of matrix units. Thus, by the universal property of $\mathcal{K}(\ell^2(G))$, there exists a $*$-homomorphism $\lambda : \mathcal{K}(\ell^2(G)) \to C_0(G) \rtimes_\alpha G$ such that $\lambda(E_{s,t}) = e_{s,t}$. By Exercise 2.10, it follows that λ is onto. Since $\mathcal{K}(\ell^2(G))$ is simple, it follows that λ is one one. Hence λ is an isomorphism. This completes the proof. $\square$

Exercise 2.11 *Consider the Hilbert space $\ell^2(G)$. For $f \in C_0(G)$, let $M(f)$ be the bounded operator on $\ell^2(G)$ defined by the equation*

$$M(f)\xi(s) := f(s)\xi(s)$$

for $\xi \in \ell^2(G)$. Show that $M : C_0(G) \to B(\ell^2(G))$ is a non-degenerate $*$-representation. Let $\lambda := \{\lambda_s\}_{s\in G}$ be the left regular representation of G on $\ell^2(G)$. Prove that (M, λ) is a covariant pair.

Use the fact that $C_0(G) \rtimes_\alpha G$ is simple to show that $M \rtimes \lambda$ implements an isomorphism between $C_0(G) \rtimes_\alpha G$ and $\mathcal{K}(\ell^2(G))$.

Let us end this section by listing out a few examples of universal C^*-algebras which have played crucial role in the development of the subject.

Cuntz algebra O_n : Let $n \geq 2$. The Cuntz algebra O_n is defined to be the universal unital C^*-algebra generated by isometries $s_1, s_2, \cdots, s_n$ such that

$$\sum_{i=1}^{n} s_i s_i^* = 1.$$

Note that $\{s_i s_i^* : i = 1, 2, \cdots, n\}$ is a family of projections which add up to 1 which is again a projection. Thus the projections $\{s_i s_i^*\}_{i=1}^n$ form a family of mutually orthogonal projections which is equivalent to saying $s_i^* s_j = 0$ if $i \neq j$. The Cuntz algebra O_n is simple.

The non-commutative torus A_θ : Let $\theta \in \mathbb{R}$. The non-commutative torus A_θ is defined to be the universal C^*-algebra generated by two unitaries u, v such that $uv = e^{2\pi i\theta} vu$. Define $R_\theta : \mathbb{T} \to \mathbb{T}$ by $R_\theta(z) = e^{-2\pi i\theta} z$. Note that R_θ is a homeomorphism of T. Consequently, this gives rise to an action of the cyclic group $\mathbb{Z}$ on $C(\mathbb{T})$. Show that $C(\mathbb{T}) \rtimes \mathbb{Z}$ is isomorphic to A_θ. If θ is irrational, then A_θ is simple.

The computation of K-theoretic invariants for the two C^*-algebras listed above were significant breakthroughs in operator K-theory. The non-commutative torus still remains one of the widely studied example in noncommutative geometry.

The odd dimensional quantum sphere : Let $0 < q < 1$ be given and $\ell \geq 0$. The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations:

$$\begin{aligned} z_i z_j &= q z_j z_i, && 1 \leq j < i \leq \ell+1, \\ z_i^* z_j &= q z_j z_i^*, && 1 \leq i \neq j \leq \ell+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2) \sum_{k>i} z_k z_k^* &= 0, && 1 \leq i \leq \ell+1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

Note that for $\ell = 0$, the C^*-algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the torus and for $\ell = 1$, it is denoted $C(SU_q(2))$. The C^*-algebra $C(SU_q(2))$ is one of the first examples in Woronowicz theory of compact quantum groups and is one of the first examples whose representation theory was explicitly worked out.

3 The Toeplitz algebra and the unilateral shift

In this section, we discuss the C^*-algebra generated by the unilateral shift on $\ell^2(\mathbb{N})$. We prove Coburn's theorem which asserts that it is the universal C^*-algebra generated by a single isometry. Coburn's theorem is a fundamental theorem and we will see its importance when we discuss Cuntz' proof of Bott periodicity in K-theory.

Definition 3.1 *Let $\mathcal{T}$ be the universal unital C^*-algebra generated by v such that $v^*v = 1$. The C^*-algebra $\mathcal{T}$ is called the Toeplitz algebra.*

By Remark 2.3, the C^*-algebra $\mathcal{T}$ exists. Consider the Hilbert space $\ell^2(\mathbb{N})$. Let $\{\delta_n : n \geq 0\}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. Let S be the bounded operator on $\ell^2(\mathbb{N})$ such that $S(\delta_n) = \delta_{n+1}$. The operator S is called *the unilateral shift* on $\ell^2(\mathbb{N})$. Clearly $S^*S = 1$. Thus, by the universal property of $\mathcal{T}$, there exists a unique $*$-homomorphism $\mathcal{T} \to C^*(S)$ which maps $v \to S$. Here $C^*(S)$ denotes the C^*-algebra generated by S. Coburn's theorem asserts that this map is indeed an isomorphism which is the main aim of this section.

Let us take a closer look at the C^*-algebra $C^*(S)$. For $m, n \geq 0$, let $E_{m,n} = \theta_{\delta_m,\delta_n}$. Set $P := 1 - SS^*$. Note that $P = E_{0,0}$ and $E_{m,n} = S^m P S^{*n}$. Since the linear span of $\{E_{m,n} : m, n \in \mathbb{N}\}$ is dense in $\mathcal{K}(\ell^2(\mathbb{N}))$, it follows that $\mathcal{K}(\ell^2(\mathbb{N}))$ is contained in $C^*(S)$. Hence $\mathcal{K}(\ell^2(\mathbb{N}))$ is an ideal in $C^*(S)$. Note that $\dot{S}$, the image of S under the canonical surjection, in the quotient $C^*(S)/\mathcal{K}$ is a unitary. Thus the quotient is generated by a single unitary $\dot{S}$

Lemma 3.2 *The spectrum of $\dot{S}$ in $C^*(S)/\mathcal{K}$ is $\mathbb{T}$.*

Proof. For $z \in \mathbb{T}$, let U_z be the unitary defined by the equation $U_z(\delta_n) = z^n\delta_n$. Note that $U_zSU_z^* = zS$ for every $z \in \mathbb{T}$. Fix $z \in \mathbb{T}$. The map $T \to U_zTU_z^*$ defines an automorphism of $C^*(S)$ which leaves the ideal $\mathcal{K}$ invariant. Thus, it descends to an automorphism, let us denote it by α_z, on the quotient $C^*(S)/\mathcal{K}$. Note that $\alpha_z(\dot{S}) = z\dot{S}$.

Denote the spectrum of $\dot{S}$ by $\sigma(\dot{S})$. Fix $z \in \mathbb{T}$. Since α_z is an automorphism, it follows that $\sigma(\alpha_z(\dot{S})) = \sigma(\dot{S})$. But $\alpha_z(\dot{S}) = z\dot{S}$. Hence $\sigma(\alpha_z(\dot{S})) = z\sigma(\dot{S})$. This implies that $\sigma(\dot{S})$ is invariant under multiplication by $\mathbb{T}$. Since $\dot{S}$ is a unitary, it follows that $\sigma(\dot{S})$ is contained in $\mathbb{T}$. Hence $\sigma(\dot{S}) = \mathbb{T}$. This completes the proof. $\square$

Denote the function $\mathbb{T} \ni z \to z \in \mathbb{C}$ by z itself. Now continuous functional calculus and 3.2 implies that there exists a $*$-homomorphism $C^*(S) \to C(\mathbb{T})$ which maps $S \to z$. Summarising our discussion, we have the following exact sequence

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow C^*(S) \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

where the map $C^*(S) \to C(\mathbb{T})$ sends $S \to z$ and the map $\mathcal{K} \to C^*(S)$ is the natural inclusion.

The first step towards the proof of Coburn's theorem is to derive a similar exact sequence for the Toeplitz algebra $\mathcal{T}$. We imitate what we did for $C^*(S)$. Set $p := 1 - vv^*$. Then $p \neq 0$. For $m, n \geq 0$, set $e_{m,n} = v^m p v^{*n}$. Note that $v^* p = 0$. Hence $v^{*n} p = 0$ for every $n \geq 1$. Taking adjoints, we get $pv^m = 0$ if $m \geq 1$. Note that $v^{*n} v^m = v^{m-n}$ if $m \geq n$ and if $m < n$ then $v^{*n} v^m = v^{*(n-m)}$.

Let $m_1, n_1, m_2, n_2 \geq 0$ be given. Suppose $n_1 > m_2$. Then

$$e_{m_1,n_1} e_{m_2,n_2} = v^{m_1} p v^{*n_1} v^{m_2} p v^{*n_2} = v^{m_1} p v^{*(n_1 - m_2)} p v^{*n_2} = 0.$$

A similar calculation reveals that if $m_2 > n_1$ then $e_{m_1,n_1} e_{m_2,n_2} = 0$. Clearly if $m_2 = n_1$, then $e_{m_1,n_1} e_{m_2,n_2} = e_{m_1,n_2}$. Hence

$$e_{m_1,n_1} e_{m_2,n_2} = \delta_{m_2,n_1} e_{m_1,n_2}.$$

Clearly $e_{m,n}^* = e_{n,m}$ for $m, n \geq 0$. Thus $e_{m,n}$ is a system of matrix units. Let I be the closed linear span of $\{e_{m,n} : m, n \geq 0\}$. Note that I is an ideal in $\mathcal{T}$. Indeed I is the ideal generated by p. By the universal property, there exists a homomorphism $\lambda : \mathcal{T} \to C(\mathbb{T})$ such that $\lambda(v) = z$. Since $C(\mathbb{T})$ is the universal C^*-algebra generated by a single v such that $v^* v = 1$ and $1 - vv^* = 0$, it follows that the kernel of λ is I.

Thus, we obtain a short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

Theorem 3.3 (Coburn) *The natural map $\mathcal{T} \to C^*(S)$ which sends $v \to S$ is an isomorphism.*

Proof. Consider the two short exact sequences which are

$$0 \longrightarrow I \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

and

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow C^*(S) \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

We have vertical arrows from the top sequence to the bottom sequence making it into a commutative diagram. Here the map $\mathcal{T} \to C^*(S)$ is the map that sends v to S and the map $C(\mathbb{T}) \to C(\mathbb{T})$ is the identity map. Now an application of the five lemma yields the proof. □

Next, we prove *the Wold decomposition* of a single isometry, a result which describes how a generic isometry looks like. Consider the isometry S with multiplicity, i.e. consider a Hilbert space $\mathcal{K}$ and look at $S \otimes 1$ on $\ell^2(\mathbb{N}) \otimes \mathcal{K}$. Suppose U is a unitary on a different Hilbert space say $\mathcal{H}_1$. Then $\begin{bmatrix} S \otimes 1 & 0 \\ 0 & U \end{bmatrix}$ is an isometry on $(\ell^2(\mathbb{N}) \otimes \mathcal{K}) \oplus \mathcal{H}_1$. The Wold decomposition asserts that every isometry, up to a unitary equivalence, is of this form.

Theorem 3.4 (Wold decomposition) *Let $\mathcal{H}$ be a separable Hilbert space and V be an isometry on $\mathcal{H}$. Then there exists Hilbert spaces $\mathcal{K}$ and $\mathcal{H}_1$, a unitary U on $\mathcal{H}_1$ and a unitary $W : \mathcal{H} \to (\ell^2(\mathbb{N}) \otimes \mathcal{K}) \oplus \mathcal{H}_1$ such that*

$$WVW^* = \begin{bmatrix} S \otimes 1 & 0 \\ 0 & U \end{bmatrix}.$$

First a lemma.

Lemma 3.5 *Let A be a C^*-algebra and $I \subset A$ be an ideal. Suppose $\pi : I \to B(\mathcal{H})$ is a non-degenerate representation. Then there exists a unique representation $\widetilde{\pi} : A \to B(\mathcal{H})$ such that $\widetilde{\pi}(x) = \pi(x)$ for $x \in I$.*

Proof. Any non-degenerate representation can be written as a direct sum of cyclic representations. A moment's thought reveals that it suffices to prove the lemma when π is cyclic. Thus let π be cyclic and $\xi \in \mathcal{H}$ be a cyclic vector, i.e. $\{\pi(x)\xi : x \in I\}$ is dense in $\mathcal{H}$. Fix $a \in A$. Calculate as follows to observe that for $x \in I$,

$$\begin{aligned}
\langle \pi(ax)\xi | \pi(ax)\xi \rangle &= \langle \pi(x^*a^*ax)\xi | \xi \rangle \\
&\leq ||a||^2 \langle \pi(x^*x)\xi | \xi \rangle \text{ (since } x^*a^*ax \leq ||a||^2 x^*x) \\
&\leq ||a||^2 \langle \pi(x)\xi | \pi(x)\xi \rangle.
\end{aligned}$$

The above calculation implies that there exists a unique bounded operator, denoted $\widetilde{\pi}(a)$ such that $\widetilde{\pi}(a)\pi(x)\xi = \pi(ax)\xi$ for every $x \in I$. If $a \in I$ then $\widetilde{\pi}(a)\pi(x)\xi = \pi(ax)\xi = \pi(a)\pi(x)\xi$ for every $x \in I$. Since $\{\pi(x)\xi : x \in I\}$ is dense in $\mathcal{H}$, it follows that $\widetilde{\pi}(a) = \pi(a)$. Evaluating on the dense set $\{\pi(x)\xi : x \in I\}$, it is routine to see that $\widetilde{\pi}$ is a $*$-representation.

Uniqueness follows from the fact that $\{\pi(x)\eta : x \in I, \eta \in \mathcal{H}\}$ is total in $\mathcal{H}$. This completes the proof. □

Proof of Theorem 3.4. Let V be an isometry on $\mathcal{H}$. By Coburn's theorem, there exists a representation $\pi : C^*(S) \to B(\mathcal{H})$ such that $\pi(S) = V$. Let $I := \mathcal{K}(\ell^2(\mathbb{N}))$. Set

$\mathcal{H}_0 = \pi(I)\mathcal{H}$ and $\mathcal{H}_1 = \mathcal{H}_0^{\perp}$. Since I is an ideal of $C^*(S)$, it follows that $\mathcal{H}_0$ and $\mathcal{H}_1$ are invariant under π. Note that $\pi(x)$ vanishes on $\mathcal{H}_1$ if $x \in I$. Thus $\pi(V)|_{\mathcal{H}_1}$ is a unitary. Set $U := \pi(V)|_{\mathcal{H}_1}$.

Restrict the representation π to I on $\mathcal{H}_0$. Then $\pi|_I$ is non-degenerate. Hence there exists a Hilbert space $\mathcal{K}$ and a unitary $W_0 : \mathcal{H}_0 \to \ell^2(\mathbb{N}) \otimes \mathcal{K}$ such that $W_0 \pi(x) W_0^* = x \otimes 1$ for $x \in \mathcal{K}(\ell^2(\mathbb{N}))$. The representation $W_0\pi(.)W_0^*$ and $x \to x \otimes 1$ are both extensions to A of the representation $W_0\pi(.)W_0$ defined on I. Hence $W_0\pi(x)W_0^* = x \otimes 1$ for every $x \in A$. Define

$$W : \mathcal{H}_0 \otimes \mathcal{H}_1 \to (\ell^2(\mathbb{N}) \otimes \mathcal{K}) \oplus \mathcal{H}_1$$

by $W = W_0 \oplus Id$. Then $WVW^* = W\pi(S)W^* = \begin{bmatrix} S \otimes 1 & 0 \\ 0 & U \end{bmatrix}$. This completes the proof.
□

Remark 3.6 *Here we have derived Wold decomposition from Coburn's theorem. We could for instance first prove Wold decomposition and derive Coburn's theorem as a corolllary. The derivation undertaken here is more operator algebraic in nature.*

We could study the continuous analogue of the Toeplitz algebra, called *the Wiener-Hopf algebra* and the continuous analogue of the Wold decomposition. However, the only proof that the author knows makes essential use of groupoid techniques and giving a proof will take us too far a field. We merely contend ourselves by describing the results.

Consider the Hilbert space $L^2(0,\infty)$. For $t \geq 0$, define $S_t : L^2(0,\infty) \to L^2(0,\infty)$ by

$$S_t(f)(s) := \begin{cases} f(s-t) & \text{if } t \geq s, \\ \\ 0 & \text{if } t < s \end{cases} \tag{3.5}$$

for $f \in L^2(0,\infty)$. Note that for every $t \geq 0$, S_t is an isometry and $S_{t_1}S_{t_2} = S_{t_1+t_2}$. Moreover the family $\{S_t\}_{t\geq 0}$ is strongly continuous, i.e. for $f \in L^2(0,\infty)$, the map $[0,\infty) \ni t \to S_t f \in L^2(0,\infty)$ is continuous. Such a family is called *an isometric representation of* $[0,\infty)$ or *a semigroup of isometries indexed by* $[0,\infty)$.

Let $\lambda := \{\lambda_t : t \in \mathbb{R}\}$ be the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. Denote the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2(0,\infty)$ by E. For $t \in \mathbb{R}$, set $W_t = E\lambda_t E$. Note that for $t \geq 0$, $W_t = V_t$ and for $t < 0$, $W_t = V_{-t}^*$. For $f \in L^1(\mathbb{R})$, let W_f be the operator on $L^2(0,\infty)$, defined by

$$W_f := \int f(t) W_t dt.$$

The operator W_f is called *the Wiener-Hopf operator* with symbol f. The Wiener-Hopf algebra, denoted $\mathcal{W}[0,\infty)$ is defined as the C^*-subalgebra of $B(L^2(0,\infty))$ generated by $\{W_f : f \in L^1(\mathbb{R})\}$.

It is not difficult to show that $\mathcal{W}([0,\infty))$ is generated by $\{W_f : f \in L^1(0,\infty)\}$. The main results are stated below.

(1) The C^*-algebra of compact operators on $L^2(0,\infty)$ is contained in $\mathcal{W}([0,\infty))$ and the quotient $\mathcal{W}([0,\infty))/\mathcal{K} \simeq C_0(\mathbb{R})$. That is, we have the following short exact sequence.
$$0 \longrightarrow \mathcal{K}(L^2(0,\infty)) \longrightarrow \mathcal{W}([0,\infty)) \longrightarrow C_0(\mathbb{R}) \longrightarrow 0.$$

(2) **Coburn's theorem:** Suppose $\{V_t : t \geq 0\}$ is a strongly continuous isometric representation on a Hilbert space $\mathcal{H}$. Then there exists a unique representation $\pi : \mathcal{W}([0,\infty)) \to B(\mathcal{H})$ such that
$$\pi(W_f) = \int f(t)V_t dt$$
for every $f \in L^1(0,\infty)$.

(3) **Wold decomposition:** Up to unitary equivalence, every isometric representation of $[0,\infty)$ is of the form $\begin{bmatrix} S_t \otimes 1 & 0 \\ 0 & U_t \end{bmatrix}$ where $\{U_t : t \in \mathbb{R}\}$ is a strongly continuous unitary representation of $\mathbb{R}$.

4 Measure theoretic preliminaries

Let X be a second countable locally compact Hausdorff topological space. Denote the Borel σ-algebra of X, i.e. the σ-algebra generated by open subsets of X by $\mathcal{B}_X$. On a locally compact space, we always consider this Borel σ-algebra. Let μ be a measure on $(X, \mathcal{B}_X)$. We say that μ is finite on compact sets if $\mu(K) < \infty$ for every compact $K \subset X$. Measures which are finite on compact sets are called *Radon measures*. Let μ be a Radon measure on $(X, \mathcal{B}_X)$. Denote the algebra of continuous complex valued functions on X with compact support by $C_c(X)$. We have the following.

(1) The measure μ is regular, i.e. for every $E \in \mathcal{B}_X$,

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) : K \subset E, K \text{ is compact}\} \\ &= \inf\{\mu(V) : E \subset V, V \text{ is open}\}.\end{aligned}$$

(2) The fact that μ is finite on compact sets implies that for $f \in C_c(X)$, f is integrable with respect to μ. Moreover, $C_c(X) \subset L^p(X)$ for every $1 \leq p \leq \infty$.

(3) The fact that μ is regular has the consequence that $C_c(X)$ is dense in $L^p(X)$ for every $1 \leq p < \infty$.

A linear functional $\phi : C_c(X) \to \mathbb{C}$ is said to be positive if $f \geq 0$ then $\phi(f) \geq 0$. Denote the set of positive linear functionals on $C_c(X)$ by $C_c(X)^*_+$. Let μ be a Radon measure on $(X, \mathcal{B}_X)$. Define $\phi_\mu : C_c(X) \to \mathbb{C}$ by

$$\phi_\mu(f) = \int f(x) d\mu(x)$$

for $f \in C_c(X)$. Then ϕ_μ is a postive linear functional. Denote the set of Radon measures by $\mathcal{M}(X)$.

Theorem 4.1 (Riesz representation theorem) *The map $\mathcal{M}(X) \ni \mu \to \phi_\mu \in C_c(X)^*_+$ is a bijection.*

Push forward measure: Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be measurable spaces. Suppose μ is a measure on $(X, \mathcal{B}_X)$ and $T : X \to Y$ is a measurable map. For $E \in \mathcal{B}_Y$, define

$$T_*\mu(E) := \mu(T^{-1}(E)).$$

Then $T_*\mu$ is a measure on $(Y, \mathcal{B}_Y)$. The measure $T_*\mu$ is called *the push forward of* μ *by* T. Keep the foregoing notation. Suppose $f : Y \to [0, \infty]$ is measurable. Then

$$\int f d(T_*\mu) = \int (f \circ T) d\mu.$$

Thus, a measurable function $f : Y \to \mathbb{C}$ is integrable if and only if $f \circ T$ is integrable and in that case

$$\int f d(T_*\mu) = \int (f \circ T) d\mu.$$

Inductive limit topology: Let X be a locally compact Hausdorff second countable topological space. Suppose (f_n) is a sequence in $C_c(X)$ and $f \in C_c(X)$. We say that $f_n \to f$ in *the inductive limit topology* if there exists a compact set $K \subset X$ such that

(1) for every $n \geq 1$, $supp(f_n) \subset K$, and

(2) the sequence $f_n \to f$ uniformly on X.

Suppose V is a topological vector space and $T : C_c(X) \to V$ is a linear map. We say that T is continuous with respect to the inductive limit topology if whenever f_n is a sequence in $C_c(X)$ which converges to f in the inductive limit topology, $T(f_n) \to T(f)$ in V.

Exercise 4.1 *Let X be a second countable locally compact Hausdorff topological space and let μ be a Radon measure on X. Prove that the functional*

$$C_c(X) \ni f \to \int f(x) d\mu(x) \in \mathbb{C}$$

is continuous with respect to the inductive limit topology.

Exercise 4.2 *Let X be a second countable locally compact Hausdorff topological space and let μ be a Radon measure on X. Prove that the "natural" map $C_c(X) \to L^p(X)$ is continuous with respect to the inductive limit topology for every $1 \leq p < \infty$.*

Haar measure: Let us now discuss the basics of Haar measure on a locally compact group. The letter G stands for a locally compact, second countable, Hausdorff topological group. The Borel σ-algebra of G is denoted $\mathcal{B}_G$. For $s \in G$, let $\sigma_s : G \to G$ be defined by $\sigma_s(t) = st$ and $\rho_s : G \to G$ be defined by $\rho_s(t) = ts$. For $s \in G$, let $L_s, R_s : C_c(G) \to C_c(G)$ be defined by

$$L_s f(t) = f(s^{-1}t)$$
$$R_s f(t) = f(ts).$$

A measure μ on $(G, \mathcal{B}_G)$ is said to be left invariant if $(\sigma_s)_*(\mu) = \mu$ for every $s \in G$. By Riesz representation theorem, a Radon measure μ is left invariant if and only if

$$\int f(s^{-1}t)d\mu(t) = \int f(t)d\mu(t)$$

for every $f \in C_c(G)$ and $s \in G$. We accept the following theorem without proof. We refer the reader to [9] for a proof.

Theorem 4.2 (Haar measure) *Let G be a second countable locally compact Hausdorff topological group. Then there exists a non-zero Radon measure μ which is left invariant. Moreover, if μ and ν are two non-zero left invariant Radon measures then there exists $c > 0$ such that $\nu = c\mu$.*

Definition 4.3 *A left invariant non-zero Radon measure on G is called a Haar measure on G.*

Note that any two Haar measures differ by a scalar. Of course, we could talk about a right Haar measure. Fix a left Haar measure μ on G.

Proposition 4.4 *If U is a non-empty open subset of G then $\mu(U) > 0$.*

Proof. Let U be a non-empty open subset of G. Suppose that $\mu(U) = 0$. By left invariance of μ, it follows that $\mu(xU) = 0$ for every $x \in G$. Note that $\{xU : x \in G\}$ is an open cover of G. If K is a compact set then there exists $x_1, x_2, \cdots, x_n \in G$ such that $K \subset \bigcup_{i=1}^{n} x_i U$. This has the implication that $\mu(K) = 0$ for every compact set K. By regularity, it follows that $\mu(E) = 0$ for every Borel set E which is a contradiction. Hence $\mu(U) > 0$. $\square$

Exercise 4.3 *Suppose $f \in C_c(G)$ is non-negative and $\int f d\mu = 0$. Prove that f is identically zero.*

For $s \in G$, let $\mu_s = (\rho_{s^{-1}})_*\mu$. Then μ_s is a left invariant Radon measure on G. Thus there exists a positive scalar $\Delta(s)$ such that $\mu_s = \Delta(s)\mu$. A moment's thought reveals that the map $G \ni s \to \Delta(s) \in (0, \infty)$ does not depend on the chosen Haar measure μ and it is in fact a homomorphism. The function Δ is called *the modular function* of the group G.

Exercise 4.4 *Let μ be a Haar measure on G. Prove that for $f \in G$,*

$$\int f(ts^{-1})d\mu(t) = \Delta(s) \int f(t)d\mu(t)$$

for $f \in C_c(G)$ and $s \in G$.

Proposition 4.5 *Fix $f \in C_c(G)$. The map $G \ni s \to L_s(f) \in C_c(G)$ and the map $G \ni s \to R_s(f) \in C_c(G)$ are continuous when $C_c(G)$ is given the inductive limit topology, i.e. if $s_n \to s$ then $L_{s_n}(f) \to L_s(f)$ and $R_{s_n}(f) \to R_s(f)$ in the inductive limit topology.*

Proof. Let s_n be a sequence in G such that $s_n \to s$. Choose a compact set L such that L contains $\{s_n : n \geq 1\} \cup \{s\}$. Denote the support of f by K. Note that for $t \in G$, $supp(L_t(f)) \subset tK$. Hence $supp(L_{s_n}(f))$ and $supp(L_s(f))$ are contained in LK and LK is compact.

Suppose $L_{s_n}(f)$ does not converge to $L_s(f)$ uniformly. Then there exists $\epsilon > 0$ and subsequences $t_{n_k} \in G$ such that

$$|L_{s_{n_k}}(f)(t_{n_k}) - L_s(f)(t_{n_k})| \geq \epsilon.$$

The above inequality implies that $t_{n_k} \in LK$. But LK is compact. By passing to a subsequence, if necessary, we can assume that t_{n_k} converges, say to t. Then $L_{s_{n_k}}(f)(t_{n_k}) = f(s_{n_k}^{-1}t_{n_k}) \to f(s^{-1}t)$. Similarly, $L_s(f)(t_{n_k}) \to f(s^{-1}t)$ which contradicts the fact that for every k

$$|L_{s_{n_k}}(f)(t_{n_k}) - L_s(f)(t_{n_k})| \geq \epsilon.$$

Hence the proof. $\square$

Exercise 4.5 *Suppose G acts on a locally compact space X on the left. For $s \in G$, let $L_s : C_c(X) \to C_c(X)$ be defined by $L_s(f) = f(s^{-1}x)$. Show that for $f \in C_c(X)$, the map $G \ni s \to L_s(f) \in C_c(X)$ is continuous where $C_c(X)$ is given the inductive limit topology.*

Show that for $f \in C_0(X)$, the map $G \ni s \to L_s(f) \in C_0(X)$ is continuous where $C_0(X)$ is given the norm topology. Here, the map $L_s : C_0(X) \to C_0(X)$ is defined in the same fashion. Hint: The inclusion $C_c(X) \to C_0(X)$ is continuous and has dense range.

Lemma 4.6 *The modular function Δ is continuous.*

Proof. Choose $f \in C_c(G)$ such that $\int f(t)d\mu(t) = 1$. Note that for $s \in G$,

$$\int R_{s^{-1}}f(t)d\mu(t) = \int f(ts^{-1})d\mu(s) = \Delta(s)\int f(t)d\mu(t) = \Delta(s).$$

The above equality together with Proposition 4.5 imply that Δ is continuous. $\square$

Remark 4.7 *A locally compact group G is called unimodular if $\Delta = 1$. Abelian groups are clearly unimodular. Compact groups are unimodular. For. if G is compact, the image $\Delta(G)$ is a compact subgroup of $(0, \infty)$ and the only compact subgroup of $(0, \infty)$ is $\{1\}$.*

For $E \in \mathcal{B}_G$, define $\overline{\mu}(E) = \mu(E^{-1})$. It is clear that $\overline{\mu}$ is a right invariant Haar measure. Note that $\overline{\mu}$ is the push forward measure of μ under the map $s \to s^{-1}$. The proof of the next proposition is taken from [8].

Proposition 4.8 *The measure $\overline{\mu}$ and μ are absolutely continuous with respect to each other. Moreover the Radon-Nikodym derivative is given by*

$$\frac{d\overline{\mu}}{d\mu}(s) = \Delta(s)^{-1}.$$

Equivalently, for $f \in C_c(G)$,

$$\int f(s^{-1})d\mu(s) = \int f(s)\Delta(s)^{-1}d\mu(s).$$

Proof. Let $I : C_c(G) \to \mathbb{C}$ be the positive linear functional defined by the equation

$$I(f) = \int f(s^{-1})\Delta(s^{-1})d\mu(s).$$

Let $f \in C_c(G)$ and $t \in G$ be given. Define $g(s) = f(s^{-1})\Delta(s^{-1})$. Calculate as follows to observe that

$$\begin{aligned}
I(L_t(f)) &= \int L_t(f)(s^{-1})\Delta(s^{-1})d\mu(s) \\
&= \int f(t^{-1}s^{-1})\Delta(s^{-1})d\mu(s) \\
&= \Delta(t)\int g(st)d\mu(s) \\
&= \int g(s)d\mu(s) \\
&= I(f).
\end{aligned}$$

Hence the measure ν associated to the linear functional I, via Riesz representation theorem, is left invariant. Hence there exists $C > 0$ such that for $f \in C_c(G)$,

$$\int f(s^{-1})\Delta(s)^{-1}d\mu(s) = C\int f(s)d\mu(s).$$

Let $f \in C_c(G)$ be given. Apply the above equality to the function $s \to f(s^{-1})\Delta(s^{-1})$ to see that

$$\int f(s)d\mu(s) = C\int f(s^{-1})\Delta(s^{-1})d\mu(s) = C^2\int f(s)d\mu(s).$$

Hence $C = 1$. This completes the proof. $\square$

Let us end this section by describing the Haar measure for a few examples of topological groups.

Example 4.9 *The Lebesgue measure on* $\mathbb{R}^d$ *is a Haar measure.*

Example 4.10 *Let* $\mathbb{T}$ *be the unit circle. Note that* $\mathbb{T}$ *is a compact group with respect to multiplication. We can identify* $C(\mathbb{T})$ *as follows:*

$$C(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} : \quad f \textit{ is continuous and } \ f(x+1) = f(x), \ \forall x \in \mathbb{R}\}.$$

Define $\phi : C(\mathbb{T}) \to \mathbb{C}$ *by the formula*

$$\phi(f) := \int_0^1 f(x)dx.$$

Then ϕ *is a positive linear functional on* $C(\mathbb{T})$*. Thus there exists a measure* μ *on* $\mathbb{T}$ *such that* $\int f d\mu = \phi(f)$ *for every* $f \in C(\mathbb{T})$*. Show that* μ *is a Haar measure.*

Example 4.11 *Let* G *be a countable discrete group. Then the counting measure on* G *is a Haar measure on* G*.*

Example 4.12 *The* ***ax+b-group:*** *Let*

$$G := \Big\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \neq 0, b \in \mathbb{R} \Big\}.$$

Show that G *is a closed subgroup of* $GL_2(\mathbb{R})$*. As a set* $G = \mathbb{R}\backslash\{0\} \times \mathbb{R}$*. Consider the two measures* $\frac{dadb}{|a|}$ *and* $\frac{dadb}{a^2}$ *on* G*. One of them is right invariant and the other is left invariant. Determine which one is right invariant and which one is left invariant. Compute the modular function and show that the group* G*, also called the* ***ax+b****-group, is not unimodular.*

5 Group C^*-algebras

Let G be an arbitrary, locally compact second countable topological group fixed for the rest of this section. Fix a Haar measure μ. We write $\int f(s)d\mu(s)$ simply as $\int f(s)ds$. Recall the following formulas: for $f \in C_c(G)$ and $t \in G$,

$$\int f(t^{-1}s)ds = \int f(s)ds$$
$$\int f(st)ds = \Delta(t)^{-1} \int f(s)ds$$
$$\int f(s^{-1})ds = \int f(s)\Delta(s^{-1})ds.$$

For $f, g \in C_c(G)$, let $f * g : G \to \mathbb{C}$ be defined by

$$f * g(s) = \int f(st)g(t^{-1})dt = \int f(t)g(t^{-1}s)ds.$$

The function $f * g$ is called *the convolution of f and g*[2]. Define an involution operation $*$ on $C_c(G)$ as follows. For $f \in C_c(G)$, let $f^* \in C_c(G)$ be defined by $f^*(s) = \Delta(s)^{-1}\overline{f(s^{-1})}$.

Exercise 5.1 *Show that for $f, g \in C_c(G)$, $f * g \in C_c(G)$. If K denotes the support of f and L denotes the support of g, prove that the support of $f * g$ is contained in KL.*

Proposition 5.1 *The space $C_c(G)$ with convolution as multiplication and $*$ as involution is a $*$-algebra.*

Proof. The proof is really a straightforward application of Fubini's theorem and the left invariance of the Haar measure. For the reader's benefit, let us verify that the convolution is associative and $*$ is anti-multiplicative. Let $f, g, h \in C_c(G)$ be given. For

[2] One could equally convolve L^1 functions. But most of the time, it suffices to work with the dense subspace $C_c(G)$

$s \in G$, calculate as follows to observe that

$$
\begin{aligned}
(f * g) * h(s) &= \int (f * g)(st)h(t^{-1})dt \\
&= \int \Big(\int f(str)g(r^{-1})dr \Big) h(t^{-1})dt \\
&= \int \Big(\int f(sr)g(r^{-1}t)dr \Big) h(t^{-1})dt \text{ (left invariance of the Haar measure)} \\
&= \int f(sr) \Big(g(r^{-1}t)h(t^{-1})dt \Big) dr \text{ (Fubini's theorem)} \\
&= \int f(sr)(g * h)(r^{-1})dr \\
&= (f * (g * h))(s).
\end{aligned}
$$

This proves that the convolution is associative. Let $f, g \in C_c(G)$ be given. For $f, g \in C_c(G)$ and $s \in G$, calculate as follows to observe that

$$
\begin{aligned}
(f * g)^*(s) &= \Delta(s)^{-1}\overline{(f * g)(s^{-1})} \\
&= \Delta(s)^{-1} \int \overline{f(s^{-1}t)g(t^{-1})}dt \\
&= \int g^*(t)f^*(t^{-1}s)dt \\
&= (g^* * f^*)(s).
\end{aligned}
$$

This completes the proof. $\square$

If G is discrete the characteristic functions $\delta_s \in C_c(G)$ and $\{\delta_s : s \in G\}$ spans $C_c(G)$. Moreover in the discrete case, the multiplication and the involution of basis elements are as follows:

$$
\begin{aligned}
\delta_s * \delta_t &= \delta_{st} \\
\delta_s^* &= \delta_{s^{-1}}.
\end{aligned}
$$

If G is not discrete the characteristic functions are no longer elements of $C_c(G)$. The trick to overcome this is to use *approximate identities* instead.

Let $\{U_n\}_{n=1}^{\infty}$ be a decreasing sequence of open sets containing the identity element e. We assume that U_n is symmetric around e, i.e. $U_n^{-1} = U_n$. Suppose that $\{U_n\}_{n=1}^{\infty}$ is a basis at e, i.e. given an open set U containing e, there exists N such that $U_N \subset U$. Note that such a sequence of open sets can always be constructed. For, G is metrisable and we can let V_n be the open ball (with respect to a metric inducing the topology of

G) of radius $\frac{1}{n}$ centered at e. Then set $U_n = V_n \cap V_n^{-1}$. For each n, choose $\phi_n \in C_c(G)$ such that $supp(\phi_n) \subset U_n$, $\phi_n^* = \phi_n$, $\phi_n \geq 0$ and $\int \phi_n(s)ds = 1$. Such a sequence $\{\phi_n\}_{n=1}^{\infty}$ is called *an approximate identity* of $C_c(G)$. The justification of the name *approximate identity* is due to the following proposition.

Proposition 5.2 *Keep the foregoing notation. For $f \in C_c(G)$, the sequence $\{\phi_n * f\}_{n=1}^{\infty}$ and the sequence $\{f * \phi_n\}_{n=1}^{\infty}$ converges to f in the inductive limit topology.*

Proof. Since ϕ_n is self-adjoint, it suffices to prove that $\phi_n * f \to f$ in the inductive limit topology. Let K be a compact neighbourhood at e. For large n, $U_n \subset K$ and consequently $supp(\phi_n * f) \subset supp(\phi_n)supp(f) \subset Ksupp(f)$. Thus $\{\phi_n * f\}_{n=1}^{\infty}$ is supported inside a common compact set.

Let $\epsilon > 0$ be given. The map $G \ni t \to L_t f \in C_c(G)$ is continuous when $C_c(G)$ is given the inductive limit topology. Thus there exists N large such that for $t \in U_N$, $||L_t f - f||_\infty \leq \epsilon$. For $s \in G$ and $n \geq N$, calculate as follows to observe that

$$\begin{aligned}
|\phi_n * f(s) - f(s)| &= \Big| \int \phi_n(t) f(t^{-1}s)dt - f(s) \Big| \\
&= \Big| \int_{U_n} \phi_n(t) f(t^{-1}s)dt - \int \phi_n(t) f(s)dt \Big| \\
&= \Big| \int_{U_n} \phi_n(t)(f(t^{-1}s) - f(s))dt \Big| \\
&\leq \int_{U_n} \phi_n(t) ||L_t f - f||_\infty dt \\
&\leq \epsilon.
\end{aligned}$$

Hence the sequence $\{\phi_n * f\}_{n=1}^{\infty}$ converges to f in the inductive limit topology. □

Exercise 5.2 *Suppose G is not discrete. Prove that $C_c(G)$ has no multiplicative identity. Hint: Use the fact that $C_c(G)$ has an approximate identity.*

For $f \in C_c(G)$, define $||f||_1 := \int |f(s)|ds$. It is clear that $||f^*||_1 = ||f||_1$ for $f \in C_c(G)$. Let $f, g \in C_c(G)$ be given. Calculate as follows to observe that

$$\begin{aligned}
\int |f * g(s)|ds &\leq \int \int |f(t)||g(t^{-1}s)|dtds \\
&= \int |f(t)| \Big(\int |g(t^{-1}s)|ds \Big) dt \\
&= \int |f(t)|||g||_1 dt (\text{ Haar measure is left invariant}) \\
&= ||f||_1 ||g||_1.
\end{aligned}$$

Hence $||f * g|| \leq ||f||_1||g||_1$ for $f, g \in C_c(G)$. In other words, $(C_c(G), || \, ||_1)$ is a normed $*$-algebra.

Definition 5.3 *Let $\mathcal{A}$ be a $*$-algebra and $|| \, ||$ be a norm on $\mathcal{A}$. We say that the pair $(\mathcal{A}, || \, ||)$ is a normed $*$-algebra if for $a, b \in \mathcal{A}$,*

$$||ab|| \leq ||a||||b||$$
$$||a^*|| = ||a||.$$

Let $(\mathcal{A}, || \, ||_1)$ be a normed $*$-algebra. The enveloping C^*-algebra of $\mathcal{A}$ is defined in the same fashion as in Section 2, the only difference here is that to define the universal C^*-seminorm, we consider only representations which are bounded w.r.t. $|| \, ||_1$. Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a representation. We say that π is bounded w.r.t. $|| \, ||_1$ if $||\pi(x)|| \leq ||x||_1$ for every $x \in \mathcal{A}$.

Thus, define a C^*-seminorm $|| \, ||$ on $\mathcal{A}$ as

$$||x|| := \sup\{||\pi(x)|| : \pi \text{ is a bounded } *\text{-representation}\}$$

for $x \in \mathcal{A}$. Suppose that $||x|| < \infty$ for every $x \in \mathcal{A}$. Let $I := \{x \in \mathcal{A} : ||x|| = 0\}$. Then $|| \, ||$ descends to a genuine C^*-norm on $\mathcal{A}/I$ and the completion of $\mathcal{A}/I$ is called the enveloping C^*-algebra of $\mathcal{A}$, denoted $C^*(\mathcal{A})$. Note that $*$-representations of $C^*(\mathcal{A})$ are in one-one correspondence with bounded representations of $\mathcal{A}$.

Definition 5.4 *The full group C^*-algebra, denoted $C^*(G)$, is defined as the enveloping C^*-algebra of $C_c(G)$.*

Of course, we need to show that $C^*(G)$ exists and is non-zero. This requires us to study bounded $*$-representations of $C_c(G)$ in more detail. Just like in the discrete setting, first we show that non-degenerate bounded $*$-representations of $C_c(G)$ are in 1-1 correspondence with strongly continuous unitary representations of G.

Definition 5.5 *Let $\mathcal{H}$ be a Hilbert space and $U : G \to B(\mathcal{H})$ be a map. We denote the image of an element s under U by U_s. We say that U is a strongly continuous unitary representation if*

(1) for $s, t \in G$, $U_sU_t = U_{st}$,

(2) for $s \in G$, U_s is a unitary, and

(3) for $\xi \in \mathcal{H}$, the map $G \ni s \to U_s\xi \in \mathcal{H}$ is continuous where $\mathcal{H}$ is given the norm topology.

Since we will not consider unitary representations that are not strongly continuous, we drop the modifying term "strongly continuous". Note that to check (3), it suffices to check for vectors ξ in a total set.

Exercise 5.3 *Show that Condtion* (3) *can be replaced by the following condition.*

$(3)'$ *for $\xi, \eta \in \mathcal{H}$, the map $G \ni s \to \langle U_s\xi|\eta\rangle \in \mathbb{C}$ is continuous.*

Example 5.6 *Consider the Hilbert space $L^2(G,\mu)$ where μ is a Haar measure. For $s \in G$, let λ_s be the unitary on $L^2(G)$ defined by the equation*

$$\lambda_s(f)(t) = f(s^{-1}t).$$

Then $\lambda := \{\lambda_s\}_{s\in G}$ is a unitary representation of G and is called the left regular representation of G. The only non-trivial thing to verify is the continuity condition.

Fix $f \in C_c(G)$. Note that the map $G \ni s \to \lambda_s(f) \in L^2(G)$ is the composite of the map $G \ni s \to \lambda_s(f) \in C_c(G)$ and the inclusion $C_c(G) \to L^2(G)$. But both these maps are continuous. Hence $G \ni s \to \lambda_s(f) \in L^2(G)$ is continuous.

For $s \in G$, let ρ_s be the unitary on $L^2(G)$ defined by the equation

$$\rho_s(f)(t) = f(ts).$$

Then $\rho := \{\rho_s\}_{s\in G}$ is a unitary representation of G and is called the right regular representation of G. Note that $\lambda(G)$ and $\rho(G)$ commutes with each other. It is a remarkable fact that the commutant of $\lambda(G)$ is the von Neumann algebra generated by $\rho(G)$. Similarly, $\rho(G)'$ is the von Neumann algebra generated by $\lambda(G)$.

To explain the correspondence between unitary representations of G and non-degenerate bounded $*$-representations of $C_c(G)$, we need to recall how to integrate operator valued functions.

Operator valued integration: Let $(X,\mathcal{B})$ be a measurable space and $\mathcal{H}$ be a separable Hilbert space. A function $f : X \to B(\mathcal{H})$ is said to be weakly measurable if for $\xi, \eta \in \mathcal{H}$, the map $X \ni x \to \langle f(x)\xi|\eta\rangle \in \mathbb{C}$ is measurable.

Lemma 5.7 *Keep the foregoing notation. Let $f : X \to B(\mathcal{H})$ be weakly measurable. Then the map $X \ni x \to ||f(x)|| \in \mathbb{C}$ is measurable.*

Proof. Let D be a countable dense subset of the unit ball of $\mathcal{H}$. Note that for $x \in X$,

$$||f(x)|| = \sup_{\xi,\eta \in D} |\langle f(x)\xi|\eta\rangle|.$$

Now the proof is complete. $\square$

Let μ be a measure on $(X, \mathcal{B})$. A weakly measurable map $f : X \to B(\mathcal{H})$ is said to be integrable w.r.t. μ if $\int ||f(x)||d\mu(x)|| < \infty$.

Proposition 5.8 *Let μ be a measure on $(X, \mathcal{B})$ and $f : X \to B(\mathcal{H})$ be integrable w.r.t. μ. Then there exists a unique bounded linear operator, denoted $\int f(x)d\mu(x)$, such that for $\xi, \eta \in \mathcal{H}$,*

$$\Big\langle \big(\int f(x)d\mu(x)\big)\xi\big|\eta\Big\rangle = \int \langle f(x)\xi|\eta\rangle d\mu(x)$$

for $\xi, \eta \in \mathcal{H}$. Also,

$$|| \int f(x)d\mu(x)|| \leq \int ||f(x)||d\mu(x).$$

Proof. Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be defined by $B(\xi, \eta) = \int \langle f(x)\xi|\eta\rangle d\mu(x)$. Then B is a bounded sesquilinear form on $\mathcal{H}$. Thus there exists a unique bounded linear operator, denote it by $\int f(x)d\mu(x)$, such that

$$\Big\langle \big(\int f(x)d\mu(x)\big)\xi\big|\eta\Big\rangle = \int \langle f(x)\xi|\eta\rangle d\mu(x)$$

for $\xi, \eta \in \mathcal{H}$. The estimate is obvious. Hence the proof. $\square$

Note the following properties about operator valued integration.

(1) Suppose $f, g : X \to B(\mathcal{H})$ are integrable. Then $\alpha f + g$ is integrable for every α and in that case

$$\int (\alpha f(x) + g(x))d\mu(x) = \alpha \int f(x)d\mu(x) + \int g(x)d\mu(x).$$

In short, the integral is linear.

(2) Suppose $f : X \to B(\mathcal{H})$ is integrable and $T \in B(\mathcal{H})$. Then the maps $x \to Tf(x)$ and $x \to f(x)T$ are integrable. Also , we have the equality $\int Tf(x)d\mu(x) = T\big(\int f(x)d\mu(x)\big)$ and $\big(\int f(x)d\mu(x)\big)T = \int f(x)Td\mu(x)$.

(3) Suppose $f : X \to B(\mathcal{H})$ is integrable. Then $X \ni x \to f(x)^* \in B(\mathcal{H})$ is integrable and

$$\int f(x)^*d\mu(x) = \big(\int f(x)d\mu(x)\big)^*.$$

Exercise 5.4 *Formulate a version of the dominated convergence theorem and prove it.*

Let us now return to the study of non-degenerate bounded $*$-representations of $C_c(G)$. Let $U : G \to B(\mathcal{H})$ be a unitary representation of G on $\mathcal{H}$. For $f \in C_c(G)$, let $\pi_U(f)$ be the bounded operator given by the equation

$$\pi_U(f) = \int f(s)U_s ds.$$

Note that $\pi_U(f)$ exists since $s \to f(s)U_s$ is weakly continuous and $s \to ||f(s)U_s|| = |f(s)|$ is integrable. Clearly, for $f \in C_c(G)$, $||\pi_U(f)|| \leq ||f||_1$.

Proposition 5.9 *Keep the foregoing notation. The map $\pi_U : C_c(G) \to B(\mathcal{H})$ is a bounded non-degenerate $*$-representation of $C_c(G)$. The representation π_U is continuous w.r.t. the inductive limit topology.*

Proof. First let us check that π_U preserves the adjoints. Let $f \in C_c(G)$ and $\xi, \eta \in \mathcal{H}$ be given. Then

$$\begin{aligned}
\langle \pi_U(f^*)\xi|\eta\rangle &= \int f^*(s)\langle U_s\xi|\eta\rangle ds \\
&= \int \Delta(s^{-1})\overline{f(s^{-1})}\langle U_s\xi|\eta\rangle ds \\
&= \int \Delta(s^{-1})\langle \xi|f(s^{-1})U_{s^{-1}}\eta\rangle ds \\
&= \int \langle \xi|f(s)U_s\eta\rangle ds \\
&= \int \overline{f(s)\langle U_s\eta|\xi\rangle} ds \\
&= \overline{\langle \pi_U(f)\eta|\xi\rangle} \\
&= \langle \pi_U(f)^*\xi|\eta\rangle.
\end{aligned}$$

Hence $\pi_U(f^*) = \pi_U(f)^*$.

Let $f, g \in C_c(G)$ and $\xi, \eta \in \mathcal{H}$ be given. Calculate as follows to observe that

$$\begin{aligned}
\langle \pi_U(f * g)\xi|\eta\rangle &= \int f * g(s)\langle U_s\xi|\eta\rangle ds \\
&= \int \Big(\int f(t)g(t^{-1}s)dt\Big)\langle U_s\xi|\eta\rangle ds \\
&= \int f(t)\Big(\int g(t^{-1}s)\langle U_s\xi|\eta\rangle ds\Big)dt \\
&= \int f(t)\Big(\int g(s)\langle U_{ts}\xi|\eta\rangle ds\Big)dt \\
&= \int f(t)\Big(\int g(s)\langle U_s\xi|U_t^*\eta\rangle ds\Big)dt \\
&= \int f(t)\langle \pi_U(g)\xi|U_t^*\eta\rangle dt \\
&= \int f(t)\langle U_t\pi_U(g)\xi|\eta\rangle dt \\
&= \langle \pi_U(f)\pi_U(g)\xi|\eta\rangle.
\end{aligned}$$

Consequently, we have $\pi_U(f*g) = \pi_U(f)\pi_U(g)$. This proves that π_U is a $*$-representation. We have already noted that π_U is bounded. The continuity w.r.t. the inductive limit topology follows as a consequence.

Let $\{\phi_n\}_{n=1}^{\infty}$ be the approximate identity constructed in Proposition 5.2. Keep the notation used in Prop. 5.2. We claim that $\pi_U(\phi_n)$ converges strongly to Id. Let $\xi \in \mathcal{H}$ be given. Suppose $\epsilon > 0$ is given. Choose N large such that for $s \in U_N$, $||U_s\xi - \xi|| \leq \epsilon$. For $\eta \in \mathcal{H}$ and $n \geq N$, calculate as follows to observe that

$$\begin{aligned}
|\langle \pi_U(\phi_n)\xi - \xi|\eta\rangle &= \Big|\int \phi_n(s)\langle U_s\xi|\eta\rangle ds - \int \phi_n(s)\langle \xi|\eta\rangle ds\Big| \\
&\leq \Big|\int \phi_n(s)\langle U_s\xi - \xi|\eta\rangle ds\Big| \\
&\leq \int \phi_n(s)||U_s\xi - \xi||||\eta||ds \\
&\leq \epsilon||\eta||.
\end{aligned}$$

Hence for $n \geq N$, $||\pi_U(\phi_n)\xi - \xi|| \leq \epsilon$. This proves that $\pi_U(\phi_n)\xi \to \xi$ for every $\xi \in \mathcal{H}$. This proves our claim. Thus π_U is non-degenerate. This completes the proof. $\square$

The representation π_U constructed in the previous proposition is called *the integrated form of* U.

Exercise 5.5 *Keep the notation of the previous proposition. Prove that*

(1) for $s \in G$ and $f \in C_c(G)$, $U_s\pi_U(f) = \pi_U(L_s f)$, and

(2) for $s \in G$, $\pi_U(L_s\phi_n) \to U_s$ in the strong operator topology.

Thus, if U and V are two unitary representations of G on the same Hilbert space then $\pi_U = \pi_V$ if and only if $U = V$.

Next we show that every non-degenerate bounded $*$-representation of $C_c(G)$ is of the form π_U for a unique unitary representation U of G. We need to invoke vector valued integration at a crucial point and it is worthwhile to digress a bit into vector valued integration. We start by recalling the Krein-Smulian theorem whose proof can be found for instance in [5].

Remark 5.10 (Krein-Smulian) *Let E be a separable Banach space and $\phi : E^* \to \mathbb{C}$ be a linear functional. Then ϕ is weak $*$-continuous if and only if ϕ is weak $*$-sequentially continuous. We refer the reader to Corollary 12.8 of [5].*

Vector valued integration: Suppose E is a separable Banach space and let $(X, \mathcal{B})$ be a measurable space.

(1) A map $f : X \to E$ is said to be *weakly measurable* if $\phi \circ f$ is measurable for every $\phi \in E^*$.

(2) Suppose $f : X \to E$ is weakly measurable. Then the map $X \ni x \to ||f(x)|| \in \mathbb{C}$ is measurable. This is because since E is separable, the unit ball of E^* w.r.t. to the weak $*$-topology is a compact metrizable space.

(3) Let μ be a measure on $(X, \mathcal{B})$ and $f : X \to E$ be a weakly measurable map. We say that f is integrable w.r.t μ if $x \to ||f(x)||$ is integrable. Suppose f is integrable. Define $F : E^* \to \mathbb{C}$ by
$$F(\phi) = \int \phi(f(x))d\mu(x).$$
An application of the Krein-Smulian theorem implies that F is weak $*$-continuous. Thus there exists a unique element, denoted $\int f(x)d\mu(x) \in E$, such that
$$\phi(\int f(x)d\mu(x)) = \int \phi(f(x))d\mu(x).$$
We call $\int f(x)d\mu(x)$, the integral of f w.r.t the measure μ. The $\int$ satisfies the usual linearilty properties and the dominated convergence theorem.

Now let μ be a Haar measure on G. Let $f, g \in C_c(G)$ be given. Consider g as an element of $L^1(G)$. Note that the map $G \ni s \to L_s(g) \in L^1(G)$ is continuous when $L^1(G)$ is given the norm topology. Consequently, the vector valued integral, in the sense explained above, $\int f(s)L_s(g)ds$ exists.

Lemma 5.11 *With the foregoing notation, we have $f * g = \int f(s)L_s(g)ds$ in $L^1(G)$.*

Proof. For $\phi \in L^\infty(G)$, let $\omega_\phi : L^1(G) \to \mathbb{C}$ be defined by $\omega_\phi(f) = \int f(s)\phi(s)ds$. The map $\phi \to \omega_\phi$ identifies $L^\infty(G)$ with the dual of $L^1(G)$. It suffices to show that for every $\phi \in L^\infty(G)$, $\omega_\phi(f * g) = \omega_\phi\Big(\int f(s)L_s(g)ds\Big)$. Fix $\phi \in L^\infty(G)$. Calculate as follows to observe that

$$\begin{aligned}
\omega_\phi(f * g) &= \int f * g(t)\phi(t)dt \\
&= \int \Big(\int f(s)g(s^{-1}t)ds)\phi(t)dt \\
&= \int f(s)\Big(\int \phi(t)g(s^{-1}t)dt\Big)ds \\
&= \int f(s)\Big(\int \phi(t)L_s(g)(t)dt\Big)ds \\
&= \int f(s)\omega_\phi(L_s(g))ds \\
&= \omega_\phi\Big(\int f(s)L_s(g)ds\Big).
\end{aligned}$$

Hence the proof. $\square$

Proposition 5.12 *Let $\pi : C_c(G) \to B(\mathcal{H})$ be a non-degenerate, bounded $*$-representation. Then there exists a unique unitary representation $U : G \to B(\mathcal{H})$ such that $\pi = \pi_U$.*

Proof. Uniqueness follows from Exercise 5.5. Note that $\{\pi(f)\xi : f \in C_c(G), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$. Let $s \in G$ be given. Note that for $f, g \in C_c(G)$, $(L_sf)^* * L_sg = f^* * g$. This has the consequence that for $\xi, \eta \in \mathcal{H}$ and $f, g \in C_c(G)$,

$$\langle \pi(L_sf)\xi | \pi(L_sg)\eta \rangle = \langle \pi(f)\xi | \pi(g)\eta \rangle.$$

Hence there exists a unique unitary operator, denote it by U_s, such that $U_s\pi(f)\xi = \pi(L_sf)\xi$ for $f \in C_c(G)$ and $\xi \in \mathcal{H}$. Evaluating on the total set $\{\pi(f)\xi : f \in C_c(G), \xi \in \mathcal{H}\}$, it is straightforward to verify that $U_sU_t = U_{st}$ for $s, t \in G$. To check that $\{U_s\}_{s \in G}$ is strongly continuous, it is sufficient to verify that for $f \in C_c(G)$ and $\xi \in \mathcal{H}$, the map

$G \ni s \to \pi(L_s f)\xi \in \mathcal{H}$ is continuous. But the last assertion follows as π is continuous with respect to the inductive limit topology (for the map $C_c(G) \ni f \to f \in L^1(G)$ is continuous).

We claim that $\pi = \pi_U$. Since $\{\pi(g)\xi : g \in C_c(G), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$, It suffices to show that for $f, g \in C_c(G)$ and $\xi, \eta \in \mathcal{H}$,

$$\langle \pi_U(f)\pi(g)\xi|\eta\rangle = \langle \pi(f)\pi(g)\xi|\eta\rangle.$$

Let $f, g \in C_c(G)$ and $\xi, \eta \in \mathcal{H}$ be given. Denote the linear extension of π to $L^1(G)$ by $\widetilde{\pi}$. Define $\omega : L^1(G) \to \mathbb{C}$ by $\omega(h) = \langle \widetilde{\pi}(h)\xi|\eta\rangle$. Calculate as follows to observe that

$$\begin{aligned}
\langle \pi(f)\pi(g)\xi|\eta\rangle &= \langle \pi(f * g)\xi|\eta\rangle \\
&= \omega(f * g) \\
&= \int f(s)\omega(L_s g)ds \quad \text{(by Lemma 5.11)} \\
&= \int f(s)\langle \pi(L_s g)\xi|\eta\rangle ds \\
&= \int f(s)\langle U_s \pi(g)\xi|\eta\rangle ds \\
&= \langle \Big(\int f(s)U_s ds)\pi(g)\xi|\eta\rangle \\
&= \langle \pi_U(f)\pi(g)\xi|\eta\rangle.
\end{aligned}$$

This completes the proof. □

Next we show that the universal C^*-norm is indeed a norm by exhibiting a faithful representation of $C_c(G)$. Let $\lambda := \{\lambda_s\}_{s\in G}$ be the left regular representation of G on $L^2(G)$. We denote the integrated form of the left regular representation by λ itself. Thus for $f \in C_c(G)$, $\lambda(f) = \int f(s)\lambda_s ds$. Let $f, g, h \in C_c(G)$ be given. Calculate as follows to observe that

$$\begin{aligned}
\langle \lambda(f)g|h\rangle &= \int f(s)\langle \lambda_s(g)|h\rangle ds \\
&= \int f(s)\Big(\int g(s^{-1}t)\overline{h(t)}dt\Big)ds \\
&= \int \overline{h(t)}\Big(\int f(s)g(s^{-1}t)ds\Big)dt \\
&= \int (f * g)(t)\overline{h(t)}dt \\
&= \langle f * g|h\rangle.
\end{aligned}$$

Hence for $f, g \in C_c(G)$, $\lambda(f)g = f * g$.

Lemma 5.13 *The representation λ is faithful.*

Proof. Let $f \in C_c(G)$ be such that $\lambda(f) = 0$. Consider an approximate identity $\{\phi_n\}_{n=1}^{\infty}$. Considered as an element of $L^2(G)$, $f * \phi_n = \lambda(f)\phi_n = 0$. Hence $f * \phi_n = 0$ in $C_c(G)$. But $f * \phi_n \to f$ in the inductive limit topology. Consequently, it follows that $f = 0$. Hence λ is faithful. $\square$

An immediate consequence of the previous lemma is that the universal C^*-seminorm on $C_c(G)$ given by

$$||f|| := \sup\{||\pi_U(f)|| : \ U \text{ is a unitary representation of } G\}$$

for $f \in C_c(G)$ is indeed a norm.

Definition 5.14 *The completion of $C_c(G)$ with respect to the universal C^*-norm is called the full group C^*-algebra and is denoted $C^*(G)$. For $f \in C_c(G)$, let*

$$||f||_{red} := ||\lambda(f)||.$$

Then $|| \ ||_{red}$ is a C^*-norm on $C_c(G)$ and its completion is called the reduced C^*-algebra of G and is denoted $C^*_{red}(G)$. Note that $C^*_{red}(G)$ is the C^*-subalgebra of $B(L^2(G))$ generated by $\{\lambda(f) : f \in C_c(G)\}$. The map $C_c(G) \ni f \to \lambda(f) \in C^*_{red}(G)$ extends to a surjective homomorphism from $C^*(G)$ onto $C^*_{red}(G)$.

Note that non-degenerate $*$-representations of $C^*(G)$ are in one-one correspondence with bounded non-degenerate $*$-representations of $C_c(G)$, which in turn is in one-one correspondence with unitary representations of G. In other words, the map

$$U \to \pi_U$$

identifies strongly continuous unitary representations of G and non-degenerate representations of $C^*(G)$. Moreover, this map preserves unitary equivalence, direct sum, irreducibility, etc..... Thus, in principle, studying the representation theory of a locally compact group is equivalent to studying the representation theory of $C^*(G)$. We prove Raikov's theorem, which asserts that every locally compact group has sufficiently many irreducible representations, as an application of this principle.

We need the following which is the corollary to Theorem 1.7.2 of [1].

Proposition 5.15 *Let A be a C^*-algebra and $a \in A$ be non-zero. Then there exists an irreducible representation π such that $\pi(a) \neq 0$. In other words, irreducible representations of A separates points of A.*

Theorem 5.16 (Raikov's theorem) *Let G be a locally compact second countable Hausdorff topological group. For every $s \in G$, with $s \neq e$, there exists an irreducible unitary representation U of G such that $U_s \neq Id$.*

Proof. Let $s \in G$ be such that $s \neq e$. Suppose, on the contrary, assume that for every irreducible unitary representation U of G, $U_s = Id$. Choose $f \in C_c(G)$ such that $L_s f \neq f$. Then there exists an irreducible representation π of $C^*(G)$, say on the Hilbert space $\mathcal{H}$, such that $\pi(L_s f) \neq \pi(f)$. Let U be the unitary representation of G such that $\pi = \pi_U$. Then U is irreducible. The equality $U_s \pi_U(f) = \pi_U(L_s f) \neq \pi_U(f)$ which implies that $U_s \neq I$. Hence the proof. $\square$

Remark 5.17 *As another application, we could derive that a finite group has only finitely many irreducible unitary representations, up to unitary equivalence. Suppose G is finite. Then $C^*(G)$ is finite dimensional. Hence $C^*(G) \simeq M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_r}(\mathbb{C})$. Consequently, $C^*(G)$ has exactly r irreducible representations.*

We end this section by identifying the C^*-algebra of an abelian group. For the rest of this section, let G be a locally compact, second countable, topological group which we assume is abelian. Note that $C^*(G)$ is commutative. First, we identify the spectrum of $C^*(G)$. Let $\chi : G \to \mathbb{T}$ be a continuous map. We say that χ is a character of G if $\chi(st) = \chi(s)\chi(t)$ for every $s, t \in G$. Denote the set of characters by $\widehat{G}$. For $\chi_1, \chi_2 \in \widehat{G}$, define $\chi_1.\chi_2 : G \to \mathbb{T}$ by the formula

$$\chi_1.\chi_2(s) = \chi_1(s)\chi_2(s)$$

for $s \in G$. Then $\chi_1.\chi_2 \in \widehat{G}$. With this multiplication $\widehat{G}$ is an abelian group. We endow $\widehat{G}$ with the topology of uniform convergence on compact sets. The convergence of nets in $\widehat{G}$ is as follows. Let (χ_i) be a net in $\widehat{G}$ and let $\chi \in \widehat{G}$ be given. Then $\chi_i \to \chi$ if and only if for every compact set $K \subset G$, the net (χ_i) converges uniformly to χ on K. Endowed with the topology of convergence on compact sets, $\widehat{G}$ is a topological group.

Set $A := C^*(G)$. Let $\chi \in \widehat{G}$ be given. Then χ is a 1-dimensional unitary representation of G. Thus, there exists a $*$-homomorphism denoted $\omega_\chi : A \to \mathbb{C}$ such that

$$\omega_\chi(f) = \int f(s)\chi(s)ds$$

for $f \in C_c(G)$.

Theorem 5.18 *With the foregoing notation, the map $\widehat{G} \ni \chi \to \omega_\chi \in \widehat{A}$ is a homeomorphism. As a consequence, it follows that $C^*(G) \simeq C_0(\widehat{G})$.*

Exercise 5.6 *Let E be a Banach space and $\{\phi_i\}$ be a bounded net in E^*. Suppose $\phi \in E^*$ and $\phi_i \to \phi$ in the weak $*$-topology. Let K be a compact set in E. Then $\phi_i \to \phi$ uniformly on K.*

Proof of Theorem 5.18. First we prove that $\chi \to \omega_\chi$ is continuous. Suppose χ_i is a net in $\widehat{G}$ and $\chi_i \to \chi \in \widehat{G}$. Since $\{\omega_{\chi_i}\}$ is uniformly bounded, it suffices to prove that for $f \in C_c(G)$, $\omega_{\chi_i}(f) \to \omega_\chi(f)$. Let $f \in C_c(G)$ be given. Denote the support of f by K. Let $\epsilon > 0$ be given. Choose i_0 such that for $i \geq i_0$, $|\chi_i(x) - \chi(x)| \leq \epsilon$ for $x \in K$. Calculate as follows to observe that for $i \geq i_0$,

$$\begin{aligned}
|\omega_{\chi_i}(f) - \omega_\chi(f)| &= |\int (f(s)\chi_i(s) - f(s)\chi(s))ds \\
&\leq \int_K |f(s)||\chi_i(s) - \chi(s)|ds \\
&\leq \epsilon ||f||_1.
\end{aligned}$$

This proves that $\omega_{\chi_i} \to \omega_\chi$. Hence the map $\chi \to \omega_\chi$ is continuous.

Let ω be a character of A. View ω as a 1-dimensional representation on the Hilbert space $\mathbb{C}$. Then there exists a unitary representation $\chi : G \to \mathcal{U}(\mathbb{C}) \simeq \mathbb{T}$ such that for $f \in C_c(G)$, $\omega(f) = \int f(s)\chi(s)ds = \omega_\chi(f)$. Since $C_c(G)$ is dense in $C^*(G)$, it follows that $\omega = \omega_\chi$. This proves that $\chi \to \omega_\chi$ is onto. The injectivity of the map follows from Exercise 5.5.

Consider a net $(\omega_{\chi_i}) \to \omega_\chi$. Then $\omega_{\chi_i\chi^{-1}} \to \omega_1$. It suffices to prove that $\chi_i\chi^{-1} \to 1$. Thus, with no loss of generality, we can assume that χ is the trivial character. Denote ω_χ by ω_0. Let K be a compact subset of G and let $\epsilon > 0$ be given. Choose $f \in C_c(G)$ such that $f \geq 0$ and $\int f(s)ds = 1$. Note that $f * \chi(s) = 1$ for every $s \in G$.

Note that the inclusion $C_c(G) \to A$ is continuous. Hence the set $\{L_{s^{-1}}f : s \in K\}$ is a compact subset of A. By Exercise 5.6, there exists i_0 such that for $i \geq i_0$ and $s \in K$,

$$|f * \chi_i(s) - \chi(s)| = |f * \chi_i(s) - f * \chi(s)| = |\omega_{\chi_i}(L_{s^{-1}}f) - \omega_0(L_{s^{-1}}f)| \leq \epsilon.$$

Choose i_1 such that for $i \geq i_1$, $|\omega_{\chi_i}(f) - \omega_0(f)| \leq \epsilon$. Choose $i_2 \geq i_0, i_1$. Let $i \geq i_2$ and

$s \in G$ be given. Calculate as follows to observe that

$$\begin{aligned}
|f * \chi_i(s) - \chi_i(s)| &= |\int f(t)\chi_i(t^{-1}s)ds - \chi_i(s)\int f(t)dt| \\
&\leq |\int \chi_i(s)\overline{(\chi_i(t)-1)}f(t)dt| \\
&\leq |\chi_i(s)|\left|\int f(t)(\chi_i(t)-1)dt\right| \\
&\leq |\omega_{\chi_i}(f) - \omega_0(f)| \\
&\leq \epsilon.
\end{aligned}$$

Combining the above two inequalities, we see that for $i \geq i_2$ and $s \in K$, $|\chi_i(s) - \chi(s)| \leq 2\epsilon$. This proves that $\chi_i \to \chi$. Hence the map $\widehat{G} \ni \chi \to \omega_\chi \in \widehat{A}$ is a homeomorphism. This completes the proof. $\square$

We end this section by discussing Plancherel's theorem for abelian groups. Let G be a locally compact second countable Hausdorff topological group which we assume is abelian. For $f \in L^1(G)$, let $\widehat{f} : \widehat{G} \to \mathbb{C}$ be defined by

$$\widehat{f}(\chi) = \int f(s)\overline{\chi(s)}ds$$

for $\chi \in \widehat{G}$. The function $\widehat{f}$ is called the Fourier transform of f. Note that for $f \in C_c(G)$, $\widehat{f}(\chi) = \omega_{\overline{\chi}}(f)$. Hence $\widehat{f} \in C_0(\widehat{G})$ for $f \in C_c(G)$. Using the fact that $C_c(G)$ is dense in $L^1(G)$, it follows at once that $\widehat{f} \in C_0(\widehat{G})$ for $f \in L^1(G)$.

Theorem 5.19 (Plancherel's theorem) *(1) For $f \in L^1(G) \cap L^2(G)$, $\widehat{f} \in L^2(\widehat{G})$.*

(2) There exists a unique Haar measure $\overline{\mu}$ on $\widehat{G}$ such that the map

$$L^1(G) \cap L^2(G) \ni f \to \widehat{f} \in L^2(\widehat{G})$$

extends to a unitary map from $L^2(G)$ onto $L^2(\widehat{G})$. The unitary $f \to \widehat{f}$ is usually denoted $\mathcal{F}$.

Let $\sigma : C^*(G) \to C_0(\widehat{G})$ be the $*$-homomorphism such that for $f \in C_c(G)$ and $\chi \in \widehat{G}$,

$$\sigma(f)(\chi) = \omega_{\overline{\chi}}(f) = \int f(s)\chi(s)ds.$$

Theorem 5.18 and the Gelfand-Naimark theorem asserts that σ is well-defined and is a $*$-isomorphism. Let $\overline{\mu}$ be a Haar measure on $\widehat{G}$ as in Plancherel's theorem. Let

$M : C_0(\widehat{G}) \to B(L^2(\widehat{G}))$ be the multiplication representation, i.e. for $f \in C_0(\widehat{G})$ the operator $M(f)$ is given by

$$M(f)\xi(\chi) = f(\chi)\xi(\chi)$$

for $\xi \in L^2(\widehat{G})$. Then M is a faithful representation. Let $\lambda : C^*(G) \to B(L^2(G))$ be the left regular representation. For $f \in C_c(G)$ and $\xi \in C_c(G)$, calculate as follows to observe that

$$\begin{aligned}
\mathcal{F}\lambda(f)\xi(\chi) &= \widehat{f * \xi}(\chi) \\
&= \int (f * \xi)(s)\chi(s)ds \\
&= \int (\int f(t)\xi(t^{-1}s)dt)\chi(s)ds \\
&= \int f(t)\chi(t)\Big(\int \xi(t^{-1}s)\chi(t^{-1}s)ds\Big)dt \\
&= \int f(t)\chi(t)\Big(\int \xi(s)\chi(s)ds\Big)dt \\
&= \int f(t)\chi(t)\widehat{\xi}(\chi)dt \\
&= \omega_{\overline{\chi}}(f)\mathcal{F}\xi(\chi) \\
&= \sigma(f)(\chi)\mathcal{F}\xi(\chi).
\end{aligned}$$

Hence $\mathcal{F}\lambda(f) = M(\sigma(f))\mathcal{F}$ for $f \in C_c(G)$. Hence $\mathcal{F}\lambda(.)\mathcal{F}^* = M \circ \sigma$. But $M \circ \sigma$ is a faithful representation of $C^*(G)$. Thus we obtain the following corollary.

Corollary 5.20 *The left regular representation $\lambda : C^*(G) \to C^*_{red}(G)$ is an isomorphism.*

We finish this section by stating the Pontraygin duality theorem. We refer the reader to Chapter 4 of [9] for a proof. Let G be a locally compact abelian group. Denote the dual group by $\widehat{G}$. For $s \in G$, let $\widehat{s} : \widehat{G} \to \mathbb{T}$ be defined by

$$\widehat{s}(\chi) = \chi(s)$$

for $\chi \in \widehat{G}$. Then $\widehat{s} \in \widehat{\widehat{G}}$. Moreover the map $G \ni s \to \widehat{s} \in \widehat{\widehat{G}}$ is continuous. Raikov's theorem implies that the map $s \to \widehat{s}$ is indeed one-one.

Theorem 5.21 (Pontraygin duality) *The map $\widehat{G} \ni s \to \widehat{s} \in \widehat{\widehat{G}}$ is a homeomorphism.*

Exercise 5.7 *In this exercise, we identify the duals of a few concrete abelian groups.*

(1) *For* $\xi \in \mathbb{R}$, *let* $\chi_\xi : \mathbb{R} \to \mathbb{T}$ *be defined by* $\chi_\xi(x) = e^{2\pi i x \xi}$. *Prove that* $\mathbb{R} \ni \xi \to \chi_\xi \in \widehat{\mathbb{R}}$ *is a homeomorphism of topological groups.*

(2) *For* $z \in \mathbb{T}$, *let* $\chi_z : \mathbb{Z} \to \mathbb{T}$ *be defined by* $\chi_z(n) = z^n$. *Show that* $\mathbb{T} \ni z \to \chi_z \in \widehat{\mathbb{Z}}$ *is a homeomorphism of topological groups.*

(3) *For* $m \in \mathbb{Z}$, *let* $\chi_m : \mathbb{T} \to \mathbb{T}$ *be defined by* $\chi_m(z) = z^m$. *Show that* $\mathbb{Z} \ni m \to \chi_m \in \widehat{\mathbb{T}}$ *is a homeomorphism of topological groups.*

(4) *Identify the duals of* $\mathbb{R}^d$, $\mathbb{Z}^d$ *and* $\mathbb{T}^d$.

6 Crossed products

In this section, we discuss the notion of crossed products of C^*-algebras associated with actions of topological groups. We will omit the proofs as we have the discussed the case of group C^*-algebras and discrete crossed products in complete detail.

Let A be a C^*-algebra and G be a locally compact second countable Hausdorff topological group. By an action of G on A, we mean a map $\alpha : G \to Aut(A)$, the image of an element s under α is denoted by α_s, such that

(1) for $s \in G$, $\alpha_s : A \to A$ is a $*$-automorphism,

(2) for $s, t \in G$, $\alpha_s \circ \alpha_t = \alpha_{st}$, and

(3) for $a \in A$, the map $G \ni s \to \alpha_s(a) \in A$ is continuous when A is given the norm topology.

The triple (A, G, α) is called a C^*-dynamical system.

Exercise 6.1 *Let (A, G, α) be a C^*-dynamical system. Prove that the map*

$$G \times A \ni (s, a) \to \alpha_s(a) \in A$$

is continuous.

Example 6.1 *Let X be a left G-space where X is a locally compact second countable Hausdorff topological space. Define for $s \in G$, $\alpha_s : C_0(X) \to C_0(X)$ by*

$$\alpha_s(f)(x) = f(s^{-1}x)$$

for $f \in C_0(X)$. Then $(C_0(X), G, \alpha)$ is a C^*-dynamical system.

Let (A, G, α) be a C^*-dynamical system. Let ds be a left Haar measure on G and Δ be the modular function of G. Consider the vector space $C_c(G, A)$, i.e.

$$C_c(G, A) := \{f : G \to A : f \text{ is continuous and compactly supported}\}.$$

Define the convolution and the involution on $C_c(G, A)$ as follows : for $f, g \in C_c(G, A)$,

$$f * g(s) = \int f(t)\alpha_t(g(t^{-1}s))dt$$
$$f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*).$$

Then $C_c(G,A)$ becomes a $*$-algebra. For $f \in C_c(G,A)$, let

$$||f||_1 := \int ||f(s)||ds.$$

With the norm defined above $C_c(G,A)$ is normed $*$-algebra. The crossed product $A \rtimes_\alpha G$ is defined as the enveloping C^*-algebra of $C_c(G,A)$.

Definition 6.2 *Let (A,G,α) be a C^*-dynamical system. By a covariant representation of (A,G,α) on a Hilbert space $\mathcal{H}$, we mean a pair (π,U) such that*

(1) π is a $*$-representation of A,

(2) $U := \{U_s\}_{s\in G}$ is a strongly continuous representation of G, and

(3) for $s \in G$ and $a \in A$, the covariance condition $U_s\pi(a)U_s^* = \pi(\alpha_s(a))$ is satisfied.

We say that (π,U) is non-degenerate if π is non-degenerate.

The following theorem characterises bounded $*$-representations of $C_c(G,A)$.

Theorem 6.3 *Let (A,G,α) be a C^*-dynamical system. Let (π,U) be a non-degenerate covariant representation of (A,G,α) on a Hilbert space $\mathcal{H}$. For $f \in C_c(G,A)$, let*

$$(\pi \rtimes U)(f) := \int \pi(f(s))U_s ds.$$

Then $\pi \rtimes U$ is a non-degenerate bounded $*$-representation of $C_c(G,A)$.

Suppose $\widetilde{\pi}$ is a bounded non-degenerate $*$-representation of $C_c(G,A)$ on a Hilbert space $\mathcal{H}$. Then there exists a unique covariant representation (π,U) of (A,G,α) on $\mathcal{H}$ which is non-degenerate such that for $f \in C_c(G,A)$,

$$\widetilde{\pi}(f) = (\pi \rtimes U)(f) = \int \pi(f(s))U_s ds.$$

Remark 6.4 *The map*

$$(\pi,U) \to \pi \rtimes U$$

establishes a bijective correspondence between non-degenerate covariant representations of (A,G,α) and non-degenerate bounded $*$-representations of $C_c(G,A)$. Moreover the above correspondence preserves direct sum, irreducibility etc.....

For $f \in C_c(G, A)$, let

$$||f|| := \sup\{||(\pi \rtimes U)(f)|| : (\pi, U) \text{ is a non-degenerate covariant representation of } (A, G, \alpha)\}.$$

We will show that $||\ ||$ is indeed a genuine C^*-norm on $C_c(G, A)$. The completion of $C_c(G, A)$ with respect to this universal norm is called the *full crossed product* and is denoted $A \rtimes G$.

Next we exhibit a concrete covariant representation of (A, G, α). Let π be a faithful non-degenerate $*$-representation of A on a Hilbert space $\mathcal{H}$. Set $\widetilde{\mathcal{H}} := L^2(G, \mathcal{H})$. Recall that $\widetilde{\mathcal{H}}$ consists of weakly measurable square integrable $\mathcal{H}$-valued functions. The inner product on $\widetilde{\mathcal{H}}$ is given by

$$\langle \xi | \eta \rangle := \int \langle \xi(s) | \eta(s) \rangle ds$$

for $\xi, \eta \in \widetilde{\mathcal{H}}$. The proof that $\widetilde{\mathcal{H}}$ is a Hilbert space is similar to the case when $\mathcal{H} = \mathbb{C}$.

For $s \in G$ and $a \in A$, let λ_s and $\widetilde{\pi}(a)$ be the bounded operators on $\widetilde{\mathcal{H}}$ defined by

$$\lambda_s \xi(t) = \xi(s^{-1}t)$$
$$\widetilde{\pi}(a)\xi(t) = \pi(\alpha_t^{-1}(a))\xi(t)$$

Then $(\widetilde{\pi}, \lambda)$ is a covariant representation of (A, G, α).

Exercise 6.2 *Show that $(\widetilde{\pi}, \lambda)$ is non-degenerate.*

Proposition 6.5 *The representation $\widetilde{\pi} \rtimes \lambda$ is a faithful representation of $C_c(G, A)$.*

Proof. Let $f \in C_c(G, A)$ be such that $(\widetilde{\pi} \rtimes \lambda)(f) = 0$. For $\xi, \eta \in C_c(G)$ and $u, v \in \mathcal{H}$, let $\xi_0(s) = \xi(s)u$ and $\eta_0(s) = \eta(s)v$. Note that $\xi_0, \eta_0 \in \widetilde{\mathcal{H}}$. Set $K(s,t) = \langle \pi(\alpha_t^{-1}(f(s)))u | v \rangle$. Calculate as follows to observe that

$$\begin{aligned}
0 &= \langle (\widetilde{\pi} \rtimes \lambda)(f)\xi_0 | \eta_0 \rangle \\
&= \int \langle \widetilde{\pi}(f(s))\lambda_s \xi_0 | \eta_0 \rangle ds \\
&= \int \Big(\int \langle (\widetilde{\pi}(f(s))\lambda_s \xi_0)(t) | \eta_0(t) \rangle dt \Big) ds \\
&= \int \Big(\int \langle \pi(\alpha_t^{-1}(f(s)))\xi(s^{-1}t)u | \eta(t)v \rangle dt \Big) ds \\
&= \int \overline{\eta(t)} \Big(\xi(s^{-1}t) K(s,t) ds \Big) dt.
\end{aligned}$$

Since η is arbitrary and $t \to \int \xi(s^{-1}t)K(s,t)ds$ is continuous, it follows that for every t,

$$\int \xi(s^{-1}t)K(s,t)ds = 0.$$

The arbitrariness of ξ implies $K(s,t) = 0$ for every s, t. Hence $K(s,e) = 0$. This implies $\langle \pi(f(s))u|v\rangle = 0$ for every s, $u, v \in \mathcal{H}$. But π is faithful and hence $f(s) = 0$ for every $s \in G$. This implies that $f = 0$ and the proof is complete. $\square$

The above proposition allows us to define the reduced C^*-norm on $C_c(G, A)$. For $f \in C_c(G, A)$, let

$$||f||_{red} = ||(\widetilde{\pi} \rtimes \lambda)(f)||.$$

The faithfulness of $\widetilde{\pi} \rtimes \lambda$ implies that $||\ ||_{red}$ is a C^*-norm on $C_c(G, A)$. Moreover for $f \in C_c(G, A)$, the reduced C^*-norm of f is atmost the full C^*-norm of f, i.e.

$$||f||_{red} \leq ||f||.$$

The completion of $C_c(G, A)$ with respect to the norm $||\ ||_{red}$ is called the *reduced crossed product* and is denoted $A \rtimes_{red} G$. There is a natural surjection from $A \rtimes G$ onto $A \rtimes_{red} G$ which need not be one-one if we don't assume amenability hypothesis.

A priori it looks as if the reduced C^*-norm depends on the chosen faithful representation π. But it is in fact independent of the chosen representation. The proof of this requires us to take a detour into the theory of Hilbert C*-modules which we undertake next.

7 Hilbert C^*-modules

Hilbert C^*-modules are analogues of Hilbert spaces where the inner product takes values in a C^*-algebra. Rieffel in his seminar paper [12] succesfully demonstrated the use of Hilbert C^*-modules to understand imprimitivity theorems due to Mackey. Kasparov's development of KK-theory utilises Hilbert C*-modules in an essential way and is now an indispensable tool in several areas of operator algebras. For more on K or KK-theory, see [3] and [10].

Let B be a C^*-algebra. Suppose E is a vector space. We say that E is a right B-module if E has a right B-action satisfying the usual consistency conditions.

Definition 7.1 *Let E be a right B-module. By a B-valued inner product on E, we mean a map $\langle\ |\ \rangle : E \times E \to B$ such that*

(1) $\langle\ |\ \rangle$ is linear in the second variable and conjugate linear in the first variable,

(2) for $b \in B$ and $x, y \in E$, $\langle x|yb\rangle = \langle x|y\rangle b$,

(3) for $x, y \in E$, $\langle x|y\rangle^* = \langle y|x\rangle$,

(4) for $x \in E$, $\langle x|x\rangle$ is a positive element of B, and

(5) if $\langle x|x\rangle = 0$ then $x = 0$.

Let E be a right B-module with a B-valued inner product. For $x \in E$, set

$$||x|| := ||\langle x|x\rangle||^{\frac{1}{2}}.$$

Proposition 7.2 (Cauchy-Schwarz inequality) *For $x, y \in E$, $||\langle x|y\rangle|| \leq ||x||||y||$.*

Proof. Let ρ be a state on B. The map $E \times E \ni (e, f) \to \rho(\langle e|f\rangle) \in \mathbb{C}$ is a semi-definite inner product on E. Applying the usual Cauchy-Schwarz inequality by taking $e = x\langle x|y\rangle$ and $f = y$, we see

$$\begin{aligned}
\rho(\langle x|y\rangle^*\langle x|y\rangle) &= \rho(\langle e|f\rangle) \\
&\leq \rho(\langle e|e\rangle)^{\frac{1}{2}}\rho(\langle f|f\rangle)^{\frac{1}{2}} \\
&\leq \rho(\langle x|y\rangle^*\langle x|x\rangle\langle x|y\rangle)^{\frac{1}{2}}\rho(\langle y|y\rangle)^{\frac{1}{2}} \\
&\leq ||x||||y||\rho(\langle x|y\rangle^*\langle x|y\rangle)^{\frac{1}{2}}.
\end{aligned}$$

On simplification, we get $\rho(\langle x|y\rangle^*\langle x|y\rangle) \leq ||x||^2||y||^2$ for every state ρ. But for a positive element $a \in B$,

$$||a|| = \sup\{\rho(a) : \rho \text{ is a state of } B\}.$$

Therefore $||\langle x|y\rangle||^2 = ||\langle x|y\rangle^*\langle x|y\rangle|| \leq ||x||^2||y||^2$. Taking square roots, we have $||\langle x|y\rangle|| \leq ||x||||y||$. The proof is complete. $\square$

Exercise 7.1 *Show that for $x \in E$ and $b \in B$, $||xb|| \leq ||x||||b||$.*

Once we have the Cauchy-Schwarz inequality, it is proved as in the Hilbert space setting that $||\ ||$ defines a norm on E.

Definition 7.3 *Let E be a right B-module with a B-valued inner product. We say that E is a Hilbert B-module if E is complete with respect to the norm $||\ ||$ where for $x \in E$,*

$$||x|| = ||\langle x|x\rangle_B||^{\frac{1}{2}}.$$

Example 7.4 *Hilbert $\mathbb{C}$-modules are just Hilbert spaces. The only difference is that now the inner product is linear in the second variable as opposed to our usual convention.*

Example 7.5 *Let B be a C^*-algebra and $E := B$. The right multiplication by B makes E into a right B-module. For $x, y \in E$, define $\langle x|y\rangle = x^*y$. Then E is a Hilbert B-module. The norm on E induced by the inner product coincides with the C^*-norm on B.*

Example 7.6 *Let B be a C^*-algebra. Set*

$$H_B := \{(b_1, b_2, \cdots,) : \sum_{n=1}^{\infty} b_n^* b_n \textit{ converges in } B\}.$$

The C^*-algebra B acts on the right by coordinatewise multiplication. For $b := (b_1, b_2, b_3, \cdots,)$ and $c := (c_1, c_2, c_3, \cdots,)$, set

$$\langle b|c\rangle := \sum_{n=1}^{\infty} b_n^* c_n.$$

Then H_B is a Hilbert B-module.

Definition 7.7 *Let E_1 and E_2 be Hilbert B-modules. Suppose $T : E_1 \to E_2$ is a map. We say that T is adjointable with adjoint T^* if there exists a map (which is necessarily unique) $T^* : E_2 \to E_1$ such that*

$$\langle Tx|y\rangle = \langle x|T^*y\rangle$$

for $x \in E_1$ and $y \in E_2$. The set of adjointable operators from E_1 to E_2 is denoted by $\mathcal{L}_B(E_1, E_2)$. When $E_1 = E_2 = E$, we write $\mathcal{L}_B(E, E)$ as $\mathcal{L}_B(E)$.

Exercise 7.2 *Let $T : E_1 \to E_2$ be an adjointable operator. Show that*

(1) T is $\mathbb{C}$-linear,

(2) the map T is B-linear, and

(3) the map $T : E_1 \to E_2$ is bounded.

Show that $\mathcal{L}_B(E_1, E_2)$ is a norm closed subspace of $B(E_1, E_2)$.

Proposition 7.8 *Let E be a Hilbert B-module. Then $\mathcal{L}_B(E)$ is a C^*-algebra.*

Proof. The only thing that requires proof is that the operator norm satisfies the C^*-identity. First note that by the Cauchy-Schwarz inequality, we have for $x \in E$,

$$||x|| = \sup\{||\langle x|y\rangle|| : ||y|| = 1\}.$$

For $T \in \mathcal{L}_B(E)$, $||T|| = \sup\{||\langle Tx|y\rangle|| : ||x|| = 1 = ||y||\}$. Thus, it is clear that $||T^*|| = ||T||$. Since the operator norm is submultiplicative, it follows that for $T \in \mathcal{L}_B(E)$, $||T^*T|| \leq ||T||^2$.

Let $T \in \mathcal{L}_B(E)$ be given. Then

$$\begin{aligned} ||T||^2 &= \sup\{||Tx||^2 : ||x|| = 1\} \\ &= \sup\{||\langle Tx|Tx\rangle|| : ||x|| = 1\} \\ &= \sup\{||\langle T^*Tx|x\rangle|| : ||x|| = 1\} \\ &\leq ||T^*T|| \text{ (by Cauchy-Schwarz inequality)}. \end{aligned}$$

The proof is now complete. □

Unlike in the case of Hilbert spaces, it is not true that bounded operators between Hilbert modules are adjointable. Here is an example.

Example 7.9 *Let $B := C[0,1]$ and $J := \{f \in B : f(0) = 0\}$. Let $E_1 = J$ and $E_2 = B$. Both E_1 and E_2 are Hilbert B-modules. Consider the inclusion $T : E_1 \to E_2$, i.e. $T(x) = x$. Then T is not adjointable. Suppose not and let S be the adjoint of T. Let $h := S(1)$. Then $h \in J$. Calculate as follows to observe that for $f \in J$,*

$$\begin{aligned} \overline{f} &= \langle T(f)|1\rangle \\ &= \langle f|S(1)\rangle \\ &= \overline{f}h. \end{aligned}$$

In other words, h is a multiplicative identity of the non-unital C^*-algebra J which is absurd.

Remark 7.10 *One needs to exercise caution while dealing with Hilbert modules. For example, it is not true that if F is a submodule of E then $E = F \oplus F^{\perp}$. Can you construct a counterexample ?*

In practice, it is essential to complete right B-modules to get genuine Hilbert modules. The following proposition helps in achieving this.

Proposition 7.11 *Let B_0 be a dense $*$-subalgebra of a C^*-algebra B. Suppose E_0 is a right B_0-module with a B_0-valued inner product. E_0 is usually called a pre-Hilbert B_0-module. Denote the completion of E_0 by E. Then the B_0-module structure on E_0 lifts uniquely to make E into a Hilbert B-module.*

Proof. The proof is routine and makes essential use of Exercise 7.1. $\square$

Let us construct the Hilbert module of interest associated to a C^*-dynamical system. Suppose (A, G, α) is a C^*-dynamical system. Let $E_0 := C_c(G, A)$. Then E_0 is a vector space. We make E_0 into a right A-module as follows. For $f \in E_0$ and $a \in A$, define

$$(f.a)(s) := f(s)\alpha_s(a).$$

The A-valued inner product on E_0 is given by

$$\langle f|g\rangle_A := (f^* * g)(e) = \int \Delta(s^{-1})\alpha_s(f(s^{-1})^*)\alpha_s(g(s^{-1}))ds.$$

Verify that E_0 is a pre-Hilbert A-module. We obtain a genuine Hilbert A-module upon completion which we denote by E.

For $a \in A$, let $i_A(a) : E_0 \to E_0$ be defined by

$$i_A(a)f(s) = af(s).$$

For $s \in G$, let $i_G(s) : E_0 \to E_0$ be defined by

$$i_G(s)f(t) = \alpha_s(f(s^{-1}t)).$$

Exercise 7.3 *For $a \in A$ and $s \in G$, show that $i_A(a)$ and $i_G(s)$ extends to bounded operator on E. We denote the extensions by the same symbols. Verify that $i_A(a)^* = i_A(a)$ and $i_G(s)^* = i_G(s^{-1})$. Prove that*

(1) the map $i_A : A \to \mathcal{L}_A(E)$ is a non-degenerate $*$-representation,

(2) the map $s \to i_G(s)$ is a strongly continuous unitary representation of G on E, and

(3) for $s \in G$ and $a \in A$, the covariance condition $i_G(s)i_A(a)i_G(s)^* = i_A(\alpha_s(a))$ is satisfied.

In short, the pair (i_A, i_G) is a covariant representation of (A, G, α) on the Hilbert A-module E.

Just like we can integrate covariant representations on a Hilbert space to obtain a representation of the crossed product on the same Hilbert space, we could do the same with Hilbert modules. To do so however requires us to discuss vector valued integration one more time.

Suppose E is a Hilbert B-module. For $x \in E$, define a seminorm $|| \, ||_x$ on $\mathcal{L}_B(E)$ by $||T||_x = ||Tx|| + ||T^*x||$. The topology on $\mathcal{L}_B(E)$ induced by the family of seminorms $\{|| \, ||_x : x \in E\}$ is called the topology of $*$-strong convergence on $\mathcal{L}_B(E)$. Let (T_i) be a net in $\mathcal{L}_B(E)$ and $T \in \mathcal{L}_B(E)$ be given. Then $T_i \to T$ in the topology of $*$-strong convergence if and only if for every $x \in X$, $T_i x \to Tx$ and $T_i^* x \to T^* x$.

Proposition 7.12 *Let X be a locally compact second countable Hausdorff topological space and μ be a Radon measure on X. Let $f : X \to \mathcal{L}_B(E)$ be continuous when $\mathcal{L}_B(E)$ is given the topology of $*$-strong convergence. Suppose that the map $X \ni x \to ||f(x)|| \in [0, \infty)$ is integrable. Then there exists a unique adjointable operator on E denoted $\int f(x)d\mu(x)$ such that for $u, v \in E$,*

$$\langle \Big(\int f(x)d\mu(x) \Big) u|v\rangle = \int \langle f(x)u|v\rangle d\mu(x).$$

Proof. Fix $u \in E$. The map $X \ni x \to f(x)u \in E$ is continuous and integrable. Define

$$\Big(\int f(x)d\mu(x) \Big) u := \int f(x)u d\mu(x).$$

The assertion follows from the defining properties of vector valued integration. $\square$

Remark 7.13 *Theorem 6.3 stays true with Hilbert spaces replaced by Hilbert modules.*

Interior tensor product: One of the most important construction in Hilbert modules is the notion of interior tensor product which we discuss next. Let E be a Hilbert B-module and F be a Hilbert C-module. Suppose $\pi : B \to \mathcal{L}_C(F)$ is a $*$-homomorphism. Think of F as a left B-module and E as a right B-module. Consider the algebraic tensor product $X := E \otimes_B F$. Note that C acts naturally on the right on X. Moreover, in X, we have the equality $eb \otimes f = e \otimes \pi(b)f$ for $e \in E$, $f \in F$ and $b \in B$. Define a C-valued semi-definite inner product on X by the formula

$$\langle e_1 \otimes f_1 | e_2 \otimes f_2 \rangle = \langle f_1 | \pi(\langle e_1 | e_2 \rangle) f_2 \rangle. \tag{7.6}$$

The fact that the inner product is positive requires a bit of work.

Lemma 7.14 *Let E be a Hilbert B-module. Suppose $T \in \mathcal{L}_B(E)$. The following are equivalent.*

(1) T is a positive element of $\mathcal{L}_B(E)$.

(2) For every $x \in E$, $\langle Tx|x \rangle \geq 0$.

Proof. Suppose (1) holds. Write $T = S^*S$ with $S \in \mathcal{L}_B(E)$. Then for $x \in E$,

$$\langle Tx|x \rangle = \langle S^*Sx|x \rangle = \langle Sx|Sx \rangle \geq 0.$$

This proves that (1) implies (2).

Now suppose that (2) holds. Note that for $x \in E$, $\langle Tx|x \rangle = \langle Tx|x \rangle^* = \langle x|Tx \rangle$. For $x, y \in E$, let $[x, y] = \langle Tx|y \rangle$ and $[x, y] = \langle x|Ty \rangle$. Both $[\ ,\ ]$ and $[\ ,\ ]'$ are sesquilinear B-valued forms and $[x, x] = [x, x]'$ for $x \in E$. By the polarisation identity, it follows that the forms $[\ ,\ ]$ and $[\ ,\ ]'$ agree. Consequently $\langle Tx|y \rangle = \langle x|Ty \rangle$. In other words, $T = T^*$.

Write $T = R - S$ with $R, S \geq 0$ and $SR = RS = 0$. For $x \in E$, calculate as follows to observe that

$$\begin{aligned} 0 &\leq \langle TSx|Sx \rangle \\ &\leq -\langle S^3x|x \rangle \\ &\leq 0 (\text{ as } S^3 \text{ is positive }). \end{aligned}$$

Hence $\langle S^3x|x \rangle = 0$ for every $x \in E$. By the polarisation identity, it follows that $\langle S^3x|y \rangle = 0$ for $x, y \in E$. Hence $S^3 = 0$ which forces $S = 0$. Thus $T = R \geq 0$. This completes the proof. □

Exercise 7.4 *Let $T \in \mathcal{L}_B(E)$ be such that $T \geq 0$. Prove that*

$$||T|| = \sup\{||\langle x|Tx \rangle|| : ||x|| = 1\}.$$

Lemma 7.15 *Let E be a Hilbert B-module and $e_1, e_2, \cdots, e_n \in E$ be given. Then the matrix $(\langle e_i|e_j \rangle)$ is a positive element of $M_n(B)$.*

Proof. Consider the Hilbert B-module $B^n := B \oplus B \oplus \cdots \oplus B$. Think of elements of B^n as column vectors. For $A \in M_n(B)$, let $L_A : B^n \to B^n$ be defined by

$$L_A(x) = Ax.$$

Then L_A is an adjointable operator on B^n and the map $M_n(B) \ni A \to L_A \in \mathcal{L}_B(B^n)$ is an injective $*$-homomorphism (Justify!).

Let $A := (\langle e_i|e_j\rangle)$. It suffices to show that L_A is positive. In view of Lemma 7.14, it suffices to show that $\langle x|L_A(x)\rangle \geq 0$. Let $x := (b_1, b_2, \cdots, b_n)^t$ be given. Note that

$$\langle x|L_A(x)\rangle = \sum_{i=1}^{n} b_i^*(\sum_{j=1}^{n}\langle e_i|e_j\rangle b_j) = \sum_{i,j}\langle e_ib_i|e_jb_j\rangle = \langle \sum_i e_ib_i|\sum_i e_ib_i\rangle \geq 0.$$

This completes the proof. $\square$

We now prove that Eq. 7.6 defines a positive semi-definite inner product. Keep the notation used in the paragraph preceding Eq. 7.6. Let $x := \sum_{i=1}^{n} e_i \otimes f_i$ be an arbitrary element in X. The representation π "amplifies naturally" to a representation of $M_n(B)$ on the Hilbert C-module $F^n := F \oplus F \oplus \cdots \oplus F$. Since $(\langle e_i|e_j\rangle)$ is a positive element in $M_n(B)$, the operator $T := (\pi(\langle e_i|e_j\rangle))$ is a positive operator on F^n. Set $f := (f_1, f_2, \cdots, f_n)^t$. Then

$$\langle x|x\rangle = \langle f|Tf\rangle \geq 0.$$

Thus $\langle\ |\ \rangle$ defines a positive semi-definite C-valued inner product on X. We mod out the null vectors and complete it to obtain a genuine Hilbert C-module. We denote the resulting C-module by $E \otimes_\pi F$. The module $E \otimes_\pi F$ is called the interior tensor product or the internal tensor product of E and F.

Proposition 7.16 *Keep the foregoing notation. Suppose $T \in \mathcal{L}_B(E)$. Then there exists a unique adjointable operator denoted $T \otimes 1$ on $E \otimes_\pi F$ such that*

$$(T \otimes 1)(e \otimes f) = Te \otimes f$$

for $e \in E$ and $f \in F$.

Proof. Let $x := \sum_{i=1}^{n} e_i \otimes f_i$ be given. We claim that

$$\sum_{i,j}\langle f_i|\pi(\langle Te_i|Te_j\rangle)f_j\rangle \leq ||T||^2 \sum_{i,j}\langle f_i|\pi(\langle e_i|e_j\rangle)f_j\rangle. \tag{7.7}$$

We leave it to the reader to convince herself that once Eq. 7.7 is established, the conclusion follows. Argue as in the previous lemma, with the aid of the next exercise, that the matrix $(\langle Te_i|Te_j\rangle) \leq ||T||^2(\langle e_i|e_j\rangle)$.

Let $A := (\pi(\langle Te_i|Te_j\rangle))$ and $B := ||T||^2(\pi(\langle e_i|e_j\rangle))$. Then A and B are adjointable operators on F^n and $A \leq B$. Set $f := (f_1, f_2, \cdots, f_n)^t$. Inequality 7.7 follows from the fact that $\langle f|Af\rangle \leq \langle f|Bf\rangle$. This completes the proof. $\square$

Exercise 7.5 *Let E be a Hilbert B-module and $T \in \mathcal{L}_B(E)$ be given. Prove that for $x \in E$, $\langle Tx|Tx\rangle \leq ||T||^2\langle x|x\rangle$.*

Hint: Write $||T||^2 - T^*T$ as S^*S for some $S \in \mathcal{L}_B(E)$.

Remark 7.17 *We have the following.*

(1) The map $\mathcal{L}_B(E) \ni T \to T \otimes 1 \in \mathcal{L}_B(E \otimes_\pi F)$ is a $*$-homomorphism. If π is injective then the map $T \to T \otimes 1$ is injective.

(2) Suppose (T_i) is a bounded net which converges to T in the $*$-strong topology. Then $T_i \otimes 1 \to T \otimes 1$ in the $*$-strong topology.

Proposition 7.16 leads us to a very important notion of induced representations due to Rieffel ([12]). The data given is as follows. Suppose E is a Hilbert B-module and let $\phi : A \to \mathcal{L}_B(E)$ be a representation. E is usually called a Hilbert A-B bimodule. Suppose π is a representation of B on a Hilbert space $\mathcal{H}$. Consider the Hilbert space $\mathcal{H}_\pi := E \otimes_\pi \mathcal{H}$. For $a \in A$, define $Ind(\pi)(a)$ on $\mathcal{H}_\pi$ by

$$Ind(\pi)(a) = \phi(a) \otimes 1.$$

Then $Ind(\pi)$ is a representation of A on the Hilbert space $\mathcal{H}_\pi$. The representation $Ind(\pi)$ is called the representation induced by π via the bimodule E.

Suppose π_1 and π_2 are representations of B on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose $T : \mathcal{H}_1 \to \mathcal{H}_2$ is an intertwiner, i.e. $T\pi_1(b) = \pi_2(b)T$. Because T commutes with the action of B, the map $1 \otimes T$ is well-defined first on the algebraic level and then extends to give a genuine adjointable operator from $E \otimes_{\pi_1} \mathcal{H}_1 \to E \otimes_{\pi_2} \mathcal{H}_2$. It is clear that $1 \otimes T$ commutes with the left action of A. Or in other words, $1 \otimes T$ intertwines $Ind(\pi_1)$ and $Ind(\pi_2)$.

Summarising the above discussion, we observe that

$$\pi \to Ind(\pi)$$

is a functor from the category of representations of B to the category of representations of A.

Remark 7.18 *We can try to seek an inverse for the above functor which leads us to the notion of Morita equivalence which we will discuss later in the course. The idea is that we can define an inverse if there is a B-A Hilbert bimodule F such that $E \otimes_B F \cong A$ and $F \otimes_A E \cong B$ as bimodules. Such bimodules exist if A and B are Morita equivalent in the sense of Rieffel (see the Section 10). Rieffel's induction then ensures that the representation theory of A and that of B are the same.*

The first remarkable result due to Rieffel is that $C_0(G/H) \rtimes G$ is Morita equivalent to $C^*(H)$ where G is a locally compact group and H is a closed subgroup of G.

Let us return back to our original motivation for considering Hilbert C^*-modules which is to establish that the reduced C^*-norm does not depend on the choice of the faithful representation. Let (A, G, α) be a C^*-dynamical system. Let $E_0 := C_c(G, A)$ be the pre-Hilbert A-module constructed earlier. The inner product and the right action of A are given by

$$\langle f|g\rangle_A := \int \Delta(t)^{-1}\alpha_t(f(t^{-1})^*)\alpha_t(g(t^{-1}))dt$$
$$(f.a)(s) := f(s)\alpha_s(a)$$

for $f, g \in E_0$ and $a \in A$. Denote the completion of E_0 by E.

Let (i_A, i_G) be the covariant representation of (A, G, α) on E as in Exercise 7.3. Recall that the formulas for i_A and i_G are given by

$$i_A(a)f(t) = af(t)$$
$$i_G(s)f(t) := \alpha_s(f(s^{-1}t))$$

for $f \in E_0$, $a \in A$ and $s \in G$.

Fix a faithful non-degenerate representation π of A on $\mathcal{H}$. Set $\widetilde{\mathcal{H}} := L^2(G, \mathcal{H})$. For $a \in A$ and $s \in G$, let $\widetilde{\pi}$ and λ_s be the bounded operators on $\widetilde{\mathcal{H}}$ defined by

$$\lambda_s\xi(t) = \xi(s^{-1}t)$$
$$\widetilde{\pi}(a)\xi(t) = \pi(\alpha_t^{-1}(a))\xi(t).$$

For $f \in C_c(G, A)$, the reduced C^*-norm is $||(\widetilde{\pi} \rtimes \lambda)(f)||$. We prove that $||f||_{red} = ||(i_A \rtimes i_G)(f)||$ and the right side does not depend on the representation π.

The trick is to use Rieffel's induction. Note that π is a representation of A on $\mathcal{H}$. Thus we can form the interior tensor product $E \otimes_\pi \mathcal{H}$ which is a Hilbert space and we show that the latter Hilbert space is identified with $\widetilde{\mathcal{H}}$ by a specific unitary. Once this identification is made then we show $i_A(a)\otimes 1 = \widetilde{\pi}(a)$ and $i_G(s)\otimes 1 = \lambda_s$. On integration, we get for $f \in C_c(G,A)$, $(i_A \rtimes i_G)(f)\otimes 1 = (\widetilde{\pi}\rtimes\lambda)(f)$. Since π is faithful , the map $\mathcal{L}_B(E) \ni T \to T\otimes 1 \in B(\widetilde{\mathcal{H}})$ is 1-1 and hence preserves the norm. Consequently, $||f||_{red} = ||(i_A \rtimes i_G)(f)||$. The verification is carried out in the next exercise.

Exercise 7.6 *Assume that G is discrete. Show that there exists a unique unitary operator $U : E\otimes_\pi \mathcal{H} \to \ell^2(G)\otimes\mathcal{H}$ such that*

$$U((a\otimes\delta_t)\otimes\xi) = \delta_t \otimes \pi(\alpha_t^{-1}(a))\xi.$$

Prove that $U(i_A(a)\otimes 1)U^* = \widetilde{\pi}(a)$ and $U(i_G(s)\otimes 1)U^* = \lambda_s$ for $a \in A$ and $s \in G$. Conclude that for $f \in C_c(G,A)$,

$$||f||_{red} = ||(i_A \rtimes i_G)(f)||.$$

Treat the topological case similarly.

8 Irreducible representations of the Heisenberg group

As an application of the material developed so far, we determine in this section, the irreducible representations of the Heisenberg group. Let $n \geq 1$ and set $H_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The group law on H_{2n+1} is defined by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \langle x_1 | y_2 \rangle).$$

Verify that H_{2n+1} together with the group law defined above is indeed a topological group.

Exercise 8.1 *Let Z be the center of H_{2n+1}. Show that $Z = \{(0,0,z) : z \in \mathbb{R}\}$.*

Proposition 8.1 (Schur's lemma) *Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation. Suppose that π is irreducible. Then the commutant $\pi(G)^{'} = \mathbb{C}$.*

Proof. Note that $\pi(G)^{'}$ is a von-Neumann algebra. The irreducibility of π implies that the only projections in $\pi(G)^{'}$ are 0 and 1. However, a von-Neumann algebra is generated by its set of projections. Consequently, $\pi(G)^{'} = \mathbb{C}$. □

Proposition 8.2 *Suppose $\pi : H_{2n+1} \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation. Assume that π is irreducible. Then there exists a unique $\lambda \in \mathbb{R}$ such that for $z \in \mathbb{R}$,*

$$\pi(0,0,z) = e^{i\lambda z}.$$

For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, set $U_x := \pi(x,0,0)$ and $V_y = \pi(0,y,0)$. Then $\{U_x\}_{x \in \mathbb{R}^n}$ and $\{V_y\}_{y \in \mathbb{R}^n}$ are strongly continuous group of unitaries such that

$$U_x V_y = e^{i\lambda \langle x|y \rangle} V_y U_x \tag{8.8}$$

for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Moreover the commutant $\{U_x, V_y : x, y \in \mathbb{R}^n\}^{'} = \mathbb{C}$.

Conversely, suppose $\{U_x\}_{x \in \mathbb{R}^n}$ and $\{V_y\}_{y \in \mathbb{R}^n}$ are strongly continuous unitary representations on a Hilbert space $\mathcal{H}$ which satisfy Eq. 8.8. Moreover suppose $\{U_x, V_y : x, y \in \mathbb{R}^n\}' = \mathbb{C}$. For $(x,y,z) \in H_{2n+1}$, set

$$\pi(x,y,z) = e^{i\lambda z} V_y U_x.$$

Then π defines an irreducible representation of H_{2n+1}.

Proof. Note that $\pi(0,0,z)$ commutes with $\pi(H_{2n+1})$ for every $z \in \mathbb{R}^n$. By Schur's lemma, it follows that $\pi(0,0,z) \in \mathbb{T}$. Since π is a strongly continuous representation, it follows that the map $\mathbb{R} \ni z \to \pi(0,0,z) \in \mathbb{T}$ is a continuous group homomorphism. Hence there exists a unique $\lambda \in \mathbb{R}$ such that $\pi(0,0,z) = e^{i\lambda z}$.

Note that in H_{2n+1},

$$(x,0,0)(0,y,0) = (0,y,0)(x,0,0)(0,0,\langle x|y\rangle).$$

Hence Eq. 8.8 is satisfied. Note that $\pi(H_{2n+1})' = \{U_x, V_y : x, y \in \mathbb{R}^n\}'$. Since π is irreducible, it follows that $\{U_x, V_y : x, y \in \mathbb{R}^n\}' = \mathbb{C}$. The proof of the converse part is routine. $\square$

Remark 8.3 *The relation 8.8 when $\lambda = 1$ is usually called the "Weyl commutation" relation.*

Definition 8.4 *Let $\mathcal{H}$ be a Hilbert space and $U := \{U_x\}_{x\in\mathbb{R}^n}$ and $V := \{V_y\}_{y\in\mathbb{R}^n}$ be strongly continuous group of unitaries. We say that the pair (U,V) is a Weyl family of unitaries with phase factor λ if Eq. 8.8 is satisfied. We call the pair (U,V) irreducible if the commutant $\{U_x, V_y : x, y \in \mathbb{R}^n\}' = \mathbb{C}$.*

In view of Prop. 8.2, the problem of determining the irreducible representations of the Heisenberg group reduces to the determination of Weyl family of unitaries which are irreducible. **Case 1:** $\lambda = 0$ In this case, a Weyl family (U,V) corresponds to two unitary representations of $\mathbb{R}^n$ which commute. Equivalently, in this case, a Weyl family corresponds to a unitary representation of the cartesian product $\mathbb{R}^n \times \mathbb{R}^n$. Thus irreducible Weyl families are precisely the irreducible representations of the abelian group $\mathbb{R}^n \times \mathbb{R}^n$ or in other words the characters of $\mathbb{R}^{2n}$. The dual of $\mathbb{R}^{2n}$ is $\mathbb{R}^{2n}$.

Let $\mu \in \mathbb{R}^{2n}$ be given. Write $\mu := (\mu_1, \mu_2)$ with $\mu_1, \mu_2 \in \mathbb{R}^n$. Set $U_x := e^{i\langle \mu_1|x\rangle}$ and $V_y := e^{i\langle \mu_2|y\rangle}$. Then (U,V) is an irreducible Weyl family of unitaries on the one dimensional Hilbert space $\mathbb{C}$ with phase factor $\lambda = 0$. Up to unitary equivalence, every such Weyl family arises this way. Moreover for distinct values of μ, the corresponding Weyl families are inequivalent.

The non-trivial case is when $\lambda \neq 0$. For the rest of our discussion, λ will be a fixed non-zero real number. The first order of business is to exhibit an irreducible Weyl family with phase factor λ. Let $\mathcal{H} := L^2(\mathbb{R}^n)$. For $x, y \in \mathbb{R}^n$, let U_x and V_y be the unitary operators on $\mathcal{H}$ defined by the following equation

$$U_x f(t) := f(t-x)$$
$$V_y f(t) := e^{-i\lambda\langle y|t\rangle} f(t)$$

Proposition 8.5 *Keep the foregoing notation. The pair (U, V) is an irreducible Weyl family with phase factor λ.*

Proof. It is routine to verify that U and V are strongly continuous unitary representations of $\mathbb{R}^n$ and they satisfy Eq. 8.8. Consider $L^\infty(\mathbb{R}^n)$ and let $M : L^\infty(\mathbb{R}^n) \to B(\mathcal{H})$ be the multiplication representation, i.e. for $\phi \in L^\infty(\mathbb{R}^n)$ and $f \in \mathcal{H}$,

$$M(\phi)f(t) = \phi(t)f(t).$$

Note that $L^\infty(\mathbb{R}^n) \ni \phi \to M(\phi) \in B(\mathcal{H})$ is continuous when $L^\infty(\mathbb{R}^n)$ is given the weak $*$-topology (after identifying $L^\infty(\mathbb{R}^n)$ with the dual of $L^1(\mathbb{R}^n)$) and when $B(\mathcal{H})$ is given the weak operator topology.

Claim: The linear span of $\{e^{-i\lambda\langle y|t\rangle} : y \in \mathbb{R}^n\}$ is weak $*$-dense in $L^\infty(\mathbb{R}^n)$. Suppose not. Then there exists a non-zero $f \in L^1(\mathbb{R}^n)$ such that for $y \in \mathbb{R}^n$,

$$\int f(t)e^{-i\lambda\langle y|t\rangle}dt = 0.$$

In other words, the Fourier transform of f is zero which in turn implies $f = 0$. This is a contradiction. This proves our claim.

Suppose $T \in B(\mathcal{H})$ is such that $TU_x = U_xT$ and $TV_y = V_yT$ for $x, y \in \mathbb{R}^n$. The density of $\{e^{-i\lambda\langle y|t\rangle} : y \in \mathbb{R}^n\}$ in $L^\infty(\mathbb{R}^n)$ implies that $TM(\phi) = M(\phi)T$ for every $\phi \in L^\infty(\mathbb{R}^n)$. It is well known that the commutant of $L^\infty(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$ (see, for instance, Theorem 2.2.1 of [1]). Hence there exists $\phi \in L^\infty(\mathbb{R}^n)$ such that $T = M(\phi)$.

Now the equation $U_xTU_x^* = T$ implies that for every $x \in \mathbb{R}^n$, $\phi(t + x) = \phi(t)$ for almost all $t \in \mathbb{R}^n$. Let $\omega_\phi : C_c(\mathbb{R}^n) \to \mathbb{R}$ be defined by

$$\omega_\phi(f) = \int f(t)\phi(t)dt.$$

To show that ϕ is a scalar, it suffices to show that ω_ϕ is a scalar multiple of the linear functional $I : C_c(\mathbb{R}^n) \to \mathbb{C}$ defined by the equation

$$I(f) = \int f(t)dt.$$

Let $g \in C_c(\mathbb{R}^n)$ be such that $I(g) = 0$. Choose $f \in C_c(\mathbb{R}^n)$ such that $\int f(t)dt = 1$.

Calculate as follows to observe that

$$\begin{aligned}
\omega_\phi(g) &= \int f(s)(\int g(t)\phi(t)dt)ds \\
&= \int f(s)(\int g(t)\phi(t-s)dt)ds \\
&= \int g(t)(\int f(s)\phi(t-s)ds)dt \\
&= \int g(t)(\int f(s)\phi(-s)ds)dt \\
&= (\int g(t)dt)(\int f(s)\phi(-s)ds) \\
&= 0.
\end{aligned}$$

Hence $Ker(I) \subset Ker(\omega_\phi)$. This shows that ω_ϕ is a scalar multiple of I. Hence ϕ is a scalar and consequently T is a scalar. This proves that the commutant $\{U_x, V_y : x, y \in \mathbb{R}^n\}' = \mathbb{C}$. The proof is now complete. □

Theorem 8.6 (Stone-von Neumann) *Let $(\widetilde{U}, \widetilde{V})$ be an irreducible Weyl family of unitaries with phase factor λ on a Hilbert space $\widetilde{\mathcal{H}}$. Denote the Weyl family constructed in Proposition 8.5 by (U, V). Then $(\widetilde{U}, \widetilde{V})$ is unitarily equivalent to (U, V). This means that there exists a unitary $T : \widetilde{\mathcal{H}} \to \mathcal{H}$ such that $T\widetilde{U}_xT^* = U_x$ and $T\widetilde{V}_yT^* = V_y$.*

The proof of Stone-von Neumann's theorem relies on the following steps.

(1) First we show that Weyl family of unitaries are in 1-1 correspondence with covariant representations of the dynamical system $(C_0(\mathbb{R}^n), \mathbb{R}^n, \alpha)$ where the action α is by translations. Moreover the correspondence respects irreducibility.

(2) Thus the problem reduces to the determination of irreducible covariant representations of $(C_0(\mathbb{R}^n), \mathbb{R}^n, \alpha)$ or in other words determining the irreducible representations of the crossed product $C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n$.

(3) Next, we show that $C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n$ is isomorphic to $\mathcal{K}(L^2(\mathbb{R}^n))$.[3] Since the algebra of compact operators has only one irreducible representation, up to unitary equivalence, the theorem follows.

For $y \in \mathbb{R}^n$, let $\alpha_y : C^*(\mathbb{R}^n) \to C^*(\mathbb{R}^n)$ be defined by

$$\alpha_y f(x) = e^{-i\lambda\langle x|y\rangle} f(x).$$

[3]For the discrete version, see Prop. 2.17.

It is routine to verify that $\alpha := \{\alpha_y\}_{y\in\mathbb{R}^n}$ is an action of $\mathbb{R}^n$ on $C^*(\mathbb{R}^n)$. Let (U,V) be a Weyl family of unitaries with phase factor λ. Denote the integrated form of U by π_U. Then (π_U, V) is a covariant representation of the dynamical system $(C^*(\mathbb{R}^n), \mathbb{R}^n, \alpha)$.

Conversely, suppose (π, V) is a non-degenerate covariant representation of $(C^*(\mathbb{R}^n), \mathbb{R}^n, \alpha)$. Let $U := \{U_x\}_{x\in\mathbb{R}^n}$ be the strongly continuous unitary representation of $\mathbb{R}^n$ whose integrated form is π. Note that (π, V) is covariant implies that for $y \in \mathbb{R}^n$, $f \in C_c(\mathbb{R}^n)$ and vectors ξ, η, we have

$$\int f(x)\langle V_y U_x V_y^* \xi|\eta\rangle dx = \int e^{-i\lambda\langle x|y\rangle} f(x)\langle U_x\xi|\eta\rangle dx.$$

Since the above equality is true for every $f \in C_c(\mathbb{R}^n)$, it follows that $V_y U_x V_y^* = e^{-i\lambda\langle x|y\rangle} U_x$ for $x, y \in \mathbb{R}^n$. In other words, it follows that (U,V) is a Weyl family of unitaries with phase factor λ.

Exercise 8.2 *Prove that the correspondence $(U,V) \to (\pi_U, V)$ preserves irreducibility.*

Since the dual of $\mathbb{R}^n$ is $\mathbb{R}^n$, it follows from Gelfand-Naimark theorem (see Theorem 5.18) that the map $\sigma : C^*(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ defined by the equation

$$\sigma(f)(s) = \int e^{i\lambda\langle s|x\rangle} f(x)dx$$

is an isomorphism. Note that for $y \in \mathbb{R}^n$ and $f \in C_c(\mathbb{R}^n)$, $\sigma(\alpha_y(f)) = L_y(\sigma(f))$ where for $g \in C_0(\mathbb{R}^n)$, $L_y(g)(s) = g(s-y)$. Thus the dynamical system $(C^*(\mathbb{R}^n), \mathbb{R}^n, \alpha)$ is isomorphic to $(C_0(\mathbb{R}^n), \mathbb{R}^n, L)$. Thus, the final step in the proof of Stone-von Neumann theorem is the fact that $C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n$ is isomorphic to $\mathcal{K}(L^2(\mathbb{R}^n))$. We prove the latter assertion for a general locally compact group.[4]

Let A be a C^*-algebra and $\mathcal{A}$ be a dense $*$-algebra of A. Suppose X is a second countable locally compact Hausdorff topological space. For $a \in \mathcal{A}$ and $f \in C_c(G)$, let $f \otimes a \in C_c(X, A)$ be defined by

$$(f \otimes a)(x) := f(x)a.$$

Proposition 8.7 *The linear span of $\{f \otimes a : f \in C_c(X), a \in \mathcal{A}\}$ is dense in $C_c(X,A)$ with respect to the inductive limit topology.*

[4]We give a proof only in the unimodular case and leave the intricacies with the modular function to the interested reader.

Proof. Let $F \in C_c(X, A)$ be given. Denote the support of F by K. Choose an open set U such that $K \subset U$ and $\overline{U}$ is compact. Fix $n \geq 1$. For $x \in K$, choose an open set $U_x \subset U$ such that for $y \in U_x$, $||F(y) - F(x)|| \leq \frac{1}{2n}$. Choose $a_x \in \mathcal{A}$ such that $||F(x) - a_x|| \leq \frac{1}{2n}$. Note that for $y \in U_x$, $||F(y) - a_x|| \leq \frac{1}{n}$. The family $\{U_x : x \in K\}$ covers K. Choose a finite subcover $\{U_{x_i} : i = 1, 2, \cdots, N\}$. For $i = 1, 2, \cdots, N$, let $a_i = a_{x_i}$. Let $\{\phi_1, \phi_2, \cdots, \phi_n\}$ be a family in $C_c(X)$ such that

(a) $supp(\phi_i) \subset U_{x_i}$ and $0 \leq \phi_i \leq 1$, and

(b) for $x \in K$, $\sum_{i=1}^{N} \phi_i(x) = 1$.

Since $\sum_{i=1}^{N} \phi > 0$ on K and K is a compact set, it follows that there exists an open subset $V \subset U$ such that $K \subset V$, $\sum_{i=1}^{N} \phi > 0$ on V and $\overline{V}$ is compact. Let $\chi \in C_c(X)$ be such that $0 \leq \chi \leq 1$, $\chi = 1$ on K and $supp(\chi) \subset V$. Set $\psi := \frac{\chi}{\sum_{i=1}^{N} \phi_i}$.

Set $F_n := \sum_{i=1}^{N} \psi\phi_i \otimes a_i$. Then $supp(F_n) \subset \overline{U}$. Let $x \in X$ be given. Calculate as follows to observe that

$$\begin{aligned}
||F(x) - F_n(x)|| &= ||\sum_{i=1}^{N} \psi(x)\phi_i(x)(F(x) - a_i)|| \\
&\leq \psi(x) \sum_{i=1}^{N} \phi_i(x)||F(x) - a_i|| \\
&\leq \frac{1}{n}\psi(x) \sum_{i=1}^{N} \phi_i(x) \quad (\text{since } ||F(x) - a_i|| \leq \frac{1}{n} \text{ if } \phi_i(x) > 0) \\
&\leq \frac{1}{n}.
\end{aligned}$$

This shows that $F_n \to F$ in the inductive limit topology and the proof is complete. ☐

Let X be a locally compact second countable Hausdorff space on which G acts on the left. For $s \in G$ and $f \in C_0(X)$, define

$$\alpha_s(f)(x) = f(s^{-1}x).$$

Then $\alpha := \{\alpha_s\}_{s \in G}$ is an action of G on $C_0(X)$. Consider the vector space $C_c(X \times G)$. The vector space $C_c(X \times G)$ has a $*$-algebra structure where the multiplication and involution are defined by

$$\begin{aligned}
F * G(x, s) &= \int F(x, t)G(t^{-1}x, t^{-1}s)dt \\
F^*(x, s) &= \Delta(s)^{-1}\overline{F(s^{-1}x, s^{-1})}.
\end{aligned}$$

for $F, G \in C_c(X \times G)$.

For $F \in C_c(X \times G)$, let $\widetilde{F} \in C_c(G, C_0(X))$ be defined by

$$\widetilde{F}(s)(x) := F(x, s).$$

The map $C_c(X \times G) \ni F \to \widetilde{F} \in C_c(G, C_0(X))$ is an embedding and preserves the $*$-algebra structure. By Prop. 8.7, it follows that $C_c(X \times G)$ is dense in $C_c(G, C_0(X))$. Consequently, $C_c(X \times G)$ is a dense $*$-subalgebra of the crossed product $C_0(X) \rtimes G$.

Theorem 8.8 *Let G be a unimodular group. The crossed product $C_0(G) \rtimes G$ is isomorphic to $\mathcal{K}(L^2(G))$.*

Proof. In view of Exercise 1.2, it suffices to exhibit a family $\{\theta_{f,g} : f, g \in C_c(G)\}$ in $C_0(G) \rtimes G$ such that

(1) for $f_1, f_2, g_1, g_2 \in C_c(G)$, $\theta_{f_1,g_1}\theta_{f_2,g_2} = \langle f_2 | g_1 \rangle \theta_{f_1,g_2}$,

(2) for $f, g \in C_c(G)$, $\theta^*_{f,g} = \theta_{g,f}$, and

(3) the linear span of $\{\theta_{f,g} : f, g \in C_c(G)\}$ is dense in $C_0(G) \rtimes G$.

For $f, g \in C_c(G)$, let $\theta_{f,g} \in C_c(X \times G)$ be defined by

$$\theta_{f,g}(x, s) = f(x)\overline{g(s^{-1}x)}.$$

Here $X = G$. It is routine to check (1) and (2). By Prop. 8.7 and the fact that the map $X \times G \ni (x, s) \to (x, s^{-1}x) \to X \times X$ is a homeomorphism, it follows that the linear span of $\{\theta_{f,g} : f, g \in C_c(G)\}$ is dense in $C_0(G) \rtimes G$. The proof is now complete. $\square$

The reader interested to know more about the history of Stone-von Neumann theorem and its role in subsequent developments in the C^*-algebra should consult the excellent essay [14] by Jonathan Rosenberg.

9 The non-commutative torus A_θ

In this section, we discuss the simplicity of the C^*-algebra, called the non-commutative torus, associated to irrational rotations on the circle. The non-commutative torus is probably the widely studied example in the field of non-commutative geometry.

Definition 9.1 *Let $\theta \in \mathbb{R}$ be given. The non-commutative torus A_θ is the universal C^*-algebra generated by two unitaries u, v such that*

$$uv = e^{2\pi i\theta} vu.$$

The reason A_θ is called the non-commutative torus is because when $\theta = 0$, A_θ is isomorphic to $C(\mathbb{T}^2)$. First we realise A_θ as a crossed product. Let $\alpha : C(\mathbb{T}) \to C(\mathbb{T})$ be defined by

$$\alpha(f)(z) = f(e^{-2\pi i\theta} z).$$

Let $\widetilde{u} := z \in C(\mathbb{T})$. Then $\alpha(\widetilde{u}) = e^{-2\pi i\theta}\widetilde{u}$. Then α is an automorphism of $C(\mathbb{T})$. The automorphism α induces an action of the cyclic group $\mathbb{Z}$ on $C(\mathbb{T})$. Consider the crossed product $C(\mathbb{T}) \rtimes \mathbb{Z}$. By the definition of the crossed product, $C(\mathbb{T}) \rtimes \mathbb{Z}$ is the universal C^*-algebra generated by a copy of $C(\mathbb{T})$ and a unitary $\widetilde{v}$ such that

$$\widetilde{v} f \widetilde{v}^* = \alpha(f)$$

for $f \in C(\mathbb{T})$. However, $C(\mathbb{T})$ is generated by $\widetilde{u}$ and the equation $\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}$ is satisfied in $C(\mathbb{T}) \rtimes \mathbb{Z}$. Consequently, there exists a surjective $*$-homomorphism $\phi : A_\theta \to C(\mathbb{T}) \rtimes \mathbb{Z}$ such that $\phi(u) = \widetilde{u}$ and $\phi(v) = \widetilde{v}$.

Note that $C(\mathbb{T})$ is the universal C^*-algebra generated by the unitary $\widetilde{u}$. Consequently, there exists a homomorphism $\pi : C(\mathbb{T}) \to A_\theta$ such that $\pi(\widetilde{u}) = u$. The relation $uv = e^{2\pi i\theta} vu$ implies that the pair (π, v) is a covariant representation of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$. Denote the homomorphism $\pi \rtimes v$ from $C(\mathbb{T}) \rtimes \mathbb{Z} \to A_\theta$ by ψ. It is clear that $\psi(\widetilde{u}) = u$ and $\psi(\widetilde{v}) = v$. Hence ψ and ϕ are inverses of each other. This proves that A_θ is isomorphic to $C(\mathbb{T}) \rtimes \mathbb{Z}$.

The main theorem of this section is that if θ is irrational then A_θ is simple, i.e. it has no non-trivial closed two sided ideals. The proof makes use of a very useful notion called conditional expectation.

Definition 9.2 *Let A be a C^*-algebra and $B \subset A$ be a C^*-subalgebra. A linear map $E : A \to B$ is called a conditional expectation of A onto B if*

(1) for $b \in B$, $E(b) = b$,

(2) for $b_1, b_2 \in B$ and $a \in A$, $E(b_1 a b_2) = b_1 E(a) b_2$, and

(3) for $a \in A$, $E(a^*a) \geq 0$.

The conditional expectation E is said to be faithful if whenever $E(a^*a) = 0$, $a = 0$.

Lemma 9.3 *Let $E : A \to B$ be a conditional expectation. Then E is contractive, i.e. for $a \in A$, $||E(a)|| \leq ||a||$.*

Proof. Consider the Hilbert B-module B. Represent B as adjointable operators on B by left multiplication. That is, for $b \in B$, let $L_b : B \to B$ be defined by $L_b(c) = bc$. Then L_b is adjointable for every $b \in B$. Moreover, $B \ni b \to L_b \in \mathcal{L}_B(B)$ is an injective $*$-homomorphism. Hence for $b \in B$, $||b|| = ||L_b||$. Suppose $a \in A$ is a positive element. Calculate, using Exercise 7.4, as follows to observe that

$$\begin{aligned}
||E(a)|| &= ||L_{E(a)}|| \\
&= \sup\{||\langle b|E(a)B\rangle|| : b \in B, ||b|| = 1\} \\
&= \sup\{||b^*E(a)b|| : b \in B, ||b|| = 1\} \\
&= \sup\{||E(b^*ab)|| : b \in B, ||b|| = 1\} \\
&\leq \sup\{||E(b^*||a||b)|| : b \in B, ||b|| = 1\} \\
&\leq ||a||.
\end{aligned}$$

For $a_1, a_2 \in A$, let $\langle a_1|a_2\rangle = E(a_1^*a_2)$. Then $\langle \ | \ \rangle$ is a B-valued semi-definite inner product. Hence by Cauchy-Schwarz inequality, we have for $b \in B$ and $a \in A$,

$$||b^*E(a)|| = ||E(b^*a)|| \leq ||E(b^*b)||^{\frac{1}{2}}||E(a^*a)||^{\frac{1}{2}} \leq ||b||||a||.$$

Let (e_λ) be an approximate identity of B. The above equation implies that $||e_\lambda E(a)|| \leq ||a||$. But $e_\lambda E(a) \to E(a)$ in B. Consequently, $||E(a)|| \leq ||a||$ for every $a \in A$. This completes the proof. $\square$

Remark 9.4 *A theorem due to Tomiyama asserts that if $E : A \to B$ is a contractive linear map such that $E(b) = b$ for every $b \in B$ then E is a conditional expectation of A onto B.*

Here is an example of a conditional expectation. Suppose A is a a C^*-algebra and $\alpha := \{\alpha_s\}$ is an action of a compact group G on A. We choose the Haar measure on G so that the G has measure 1. The fixed point algebra of α, denoted A^α, is defined by

$$A^\alpha := \{a \in A : \alpha_s(a) = a \forall s \in G\}.$$

Note that A^α is a C^*-subalgebra of A. Define $E : A \to A$ by

$$E(a) = \int \alpha_s(a) ds.$$

Exercise 9.1 *Verify that E is a faithful conditional expectation of A onto A^α.*

Let (A, G, α) be a C^*-dynamical system. Assume that G is discrete and abelian. Note that $\widehat{G}$ is a compact group. For the sake of simplicity, assume that A is unital. Recall that the crossed product $A \rtimes_\alpha G$ is the universal C^*-algebra generated by a copy of A and unitaries $\{u_s : s \in G\}$ such that $u_s u_t = u_{st}$ and $u_s a u_s^* = \alpha_s(a)$ for $a \in A$. This universal picture reveals that for $\chi \in \widehat{G}$, there exists a unique $*$-homomorphism $\beta_\chi : A \rtimes G \to A \rtimes G$ such that

$$\begin{aligned} \beta_\chi(a) &= a \\ \beta_\chi(u_s) &= \chi(s) u_s. \end{aligned}$$

Then $\beta := \{\beta_\chi\}_{\chi \in \widehat{G}}$ is an action of $\widehat{G}$ on the crossed product $A \rtimes G$. The action β is called the dual action on the crossed product $A \rtimes G$. We claim that the fixed point algebra of β is A. Set $B = A \rtimes G$. It is clear that $A \subset B^\beta$. Let $E : B \to B^\beta$ be the conditional expectation given by

$$E(b) := \int \beta_\chi(b) d\chi.$$

It suffices to show that $E(b) \in A$ for every $b \in B$. Since A is closed in $A \rtimes G$ and E is continuous, it suffices to show that $E(b) \in A$ whenever b is the form $b = \sum_{s \in G} a_s u_s$. It is clear that it suffices to show that $E(au_s) = 0$ if $s \neq e$. Let $s \neq e$ and $a \in A$ be given. Note that

$$E(au_s) = \int a\chi(s) u_s d\chi = (\int \chi(s) d\chi) au_s.$$

Since $s \neq e$, by Raikov's theorem, there exists a character χ_0 of G such that $\chi_0(s) \neq 1$. Calculate as follows to observe that

$$\begin{aligned}\int \chi(s) d\chi &= \int (\chi_0\chi)(s) d\chi \quad \text{(by the left invariance of the Haar measure)} \\ &= \int \chi_0(s)\chi(s) d\chi \\ &= \chi_0(s)(\int \chi(s) d\chi).\end{aligned}$$

Since $\chi_0(s) \neq 1$, it follows that $\int \chi(s) d\chi = 0$. This proves that $E(au_s) = 0$ if $s \neq e$. Hence $A = B^\beta$.

Remark 9.5 *If $b = \sum_{s \in G} a_s u_s$ then $E(b) = a_e$.*

Let us turn our attention back to A_θ. For the rest of this section, assume that θ is irrational. Write $A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}$. Denote the generating unitary of $C(\mathbb{T})$ by u and the unitary corresponding to the generator 1 of the group $\mathbb{Z}$ by v. Then $uv = e^{2\pi i\theta} vu$. Let $E : A_\theta \to C(\mathbb{T})$ be the conditional expectation constructed out of the dual action of $\widehat{\mathbb{Z}} = \mathbb{T}$ on A_θ. For $n \geq 1$, let $E_n : A_\theta \to A_\theta$ be defined by

$$E_n(x) := \frac{1}{n+1} \sum_{k=0}^{n} u^k x u^{*k}.$$

The crucial fact that we need to conclude the simplicity of A_θ is the following.

Lemma 9.6 *For $x \in A_\theta$, $E_n(x) \to E(x)$.*

Proof. It suffices to check that for $x = u^r v^s$ with $r, s \in \mathbb{Z}$, $E_n(x) \to E(x)$. For the sequence $\{E_n\}_{n \geq 1}$ is bounded. Let $r, s \in \mathbb{Z}$ be given and let $x = u^r v^s$. It is clear that if $s = 0$, $E_n(x) = u^r = E(x)$. It suffices to consider the case when $s \neq 0$ which we henceforth assume. Set $z = e^{2\pi i s\theta}$. Since θ is irrational, it follows that $z \neq 1$. Now a simple calculation using the relation $u^k v^s = e^{2\pi i k s\theta} v^s u^k$ implies that

$$E_n(x) = \frac{1}{n+1} (\sum_{k=0}^{n} e^{2\pi i k s\theta}) u^r v^s = \frac{1}{n+1} \Big(\frac{1 - z^{n+1}}{1 - z}\Big) u^r v^s.$$

Thus, as $n \to \infty$, $E_n(x) \to 0 = E(x)$. This completes the proof. □

Theorem 9.7 *Let θ be an irrational number. The C^*-algebra A_θ is simple.*

Proof. Let $I \subset A_\theta$ be a closed 2-sided non-zero ideal of A_θ. Denote $E(I)$ be J. A consequence of the previous lemma is that $J \subset I$. Note that $J = I \cap C(\mathbb{T})$. Since E is faithful, it follows that J is non-zero. Moreover the fact that E is a conditional expectation implies that J is a two sided ideal of $C(\mathbb{T})$.

For $x \in J$, $\alpha^k(x) = v^k x v^{*k} \in I$. Clearly $\alpha^k(x) \in C(\mathbb{T})$. Thus J is an α-invariant ideal of $C(\mathbb{T})$. In other words, $\alpha(J) = J$. Let $F \subset \mathbb{T}$ be a closed subset such that

$$J = \{f \in C(\mathbb{T}) : f \text{ vanishes on } F\}.$$

The fact that $\alpha(J) = J$ implies that $e^{2\pi i k\theta}F = F$ for every $k \in \mathbb{Z}$. It is well known that for every x_0, $\{e^{2\pi i k\theta}x_0 : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. We claim that F is empty. Suppose not. Since F is closed and $e^{2\pi i k\theta}F = F$, we have $F = \mathbb{T}$. This however forces $J = 0$ which is a contradiction. Hence $F = \emptyset$. Consequently, $J = C(\mathbb{T}) \subset I$. But then the ideal generated by $C(\mathbb{T})$ is A_θ. Hence $I = A_\theta$. This completes the proof. $\square$

10 Mackey's imprimitivity theorem : the discrete case

This section is devoted to a discussion on Mackey's imprimitivity theorem cast in Rieffel's language of Hilbert C^*-modules. We only give a proof in the discrete setting and refer the reader to the monographs [11] and [17] for the topological case. In Rieffel's language, Mackey's imprimitivity theorem reads as follows.

Theorem 10.1 *Let G be a second countable locally compact topological group and H be a closed subgroup of G. The crossed product $C_0(G/H) \rtimes G$ is Morita equivalent to $C^*(H)$.*

First we proceed towards defining the notion of strong Morita equivalence due to Rieffel.

Definition 10.2 *Let A and B be C^*-algebras. An A-B imprimitivity bimodule is a vector space E which has a left A-action and a right B-action together with A-valued and B-valued inner products satisfying the following*

(1) the A-valued inner product is linear in the first variable and conjugate linear in the second variable,

(2) the B-valued inner product is linear in the second variable and conjugate linear in the first variable,

(3) for $x, y \in E$ and $a \in A$, $\langle ax|y\rangle_B = \langle x|a^*y\rangle_B$,

(4) for $x, y \in E$ and $b \in B$, $\langle xb|y\rangle_A = \langle x|yb^*\rangle_A$,

(5) for $x, y, z \in E$, $\langle x|y\rangle_A z = x\langle y|z\rangle_B$,

(6) the linear span of $\{\langle x|y\rangle_A : x, y \in E\}$ is dense in A,

(7) the linear span of $\{\langle x|y\rangle_B : x, y \in E\}$ is dense in B, and

(8) E is complete with respect to the norm induced by both the A-valued and the B-valued inner products.

Let E be an A-B imprimitivity bimodule. For $x \in E$, define

$$||x||_A := ||\langle x|x\rangle_A||^{\frac{1}{2}}$$
$$||x||_B := ||\langle x|x\rangle_B||^{\frac{1}{2}}.$$

Proposition 10.3 *With the foregoing notation, we have $||x||_A = ||x||_B$ for every $x \in E$.*

Proof. View E as a Hilbert B-module. For $a \in A$, let $L_a : E \to E$ be defined by $L_a(x) = a.x$. Then for every $a \in A$, L_a is adjointable and the map $A \ni a \to L_a \in \mathcal{L}_B(E)$ is a $*$-homomorphism. We claim that $a \to L_a$ is injective. Suppose $L_a = 0$. Then $\langle ax|y\rangle_A = 0$ for every $x, y \in E$. Consequently $a\langle x|y\rangle_A = 0$. But $\{\langle x|y\rangle_A : x, y \in E\}$ is dense in A. Hence $ab = 0$ for every $b \in A$. This shows that $a = 0$. This proves our claim.

Let $x \in E$ be given. Define $\theta_{x,x} : E \to E$ by $\theta_{x,x}(y) = x\langle x|y\rangle_B$. Note that $\theta_{x,x} = L_{\langle x|x\rangle_A}$. Thus to complete the proof, it suffices to show that for $x \in E$,

$$||\theta_{x,x}|| = ||x||_B^2.$$

It follows from Cauchy-Schwarz inequality that $||\theta_{x,x}|| \leq ||x||_B^2$. Set $y := \frac{x}{||x||_B}$ and calculate as follows to observe that

$$\begin{aligned}
||\theta_{x,x}(y)||^2 &= ||\langle x\langle x|y\rangle|x\langle x|y\rangle\rangle|| \\
&= \frac{1}{||x||^2}||\langle x|x\rangle\langle x|x\rangle\langle x|x\rangle|| \\
&= \frac{1}{||x||^2}||x||^6 \\
&= ||x||^4.
\end{aligned}$$

Hence $||x||_B^2 \leq ||\theta_{x,x}||$. Consequently, $||\theta_{x,x}|| = ||x||_B^2$. This completes the proof. □

Definition 10.4 *Let A and B be C^*-algebras. We say that A and B are Morita equivalent if there exists an A-B imprimitivity bimodule.*

Example 10.5 *Let A be a C^*-algebra. Set $E := A$ and $B := A$. Then E is a Hilbert B-module. The C^*-algebra A acts on E by left multiplication. Define an A-valued (left) inner product on E by*

$$\langle x|x\rangle_A = xy^*.$$

Then E is an A-A imprimitivity bimodule.

Example 10.6 *Let A be a C^*-algebra and $p \in A$ be a projection. Suppose the ideal generated by p is A. Let $B := pAp$. Set $E := pA$. Define a B-valued inner product on E by*

$$\langle x|y\rangle = xy^*.$$

Then E is a B-A imprimitivity bimodule.

The algebra of compact operators of a Hilbert module: Let E be a Hilbert B-module. For $x, y \in E$, let $\theta_{x,y} : E \to E$ be defined by

$$\theta_{x,y}(z) = x\langle y|z\rangle.$$

Note that for $x, y \in E$, $\theta_{x,y}$ is an adjointable operator and $\theta_{x,y}^* = \theta_{y,x}$. Moreover for $T \in \mathcal{L}_B(E)$, $T\theta_{x,y} = \theta_{Tx,y}$ and $\theta_{x,y}T = \theta_{x,T^*y}$. The C^*-algebra of compact operators on E, denoted $\mathcal{K}_B(E)$, is defined to be the closed linear span of $\{\theta_{x,y} : x, y \in E\}$.

Remark 10.7 *Let A be a C^*-algebra and consider the Hilbert A-module $E := A$. For $a \in A$, let $L_a : E \to E$ be defined by $L_a(x) = ax$. Note that for $x, y \in E$, $\theta_{x,y} = L_{xy^*}$. This implies that $A \ni a \to L_a \in \mathcal{K}_B(E)$ is an isomorphism.*

Suppose A is unital. Then L_1 is the identity operator which is not compact in the sense of Banach space theory unless the algebra A is finite dimensional. Thus compact operators in the sense of Hilbert C^*-modules need not be compact in the usual Banach space theory sense.

Exercise 10.1 *Let $E := A^n$ be n copies of A. Show that $M_n(A)$ is isomorphic to $\mathcal{K}_A(A^n)$.*

Let E be a Hilbert B-module and set $A := \mathcal{K}_B(E)$. The C^*-algebra A acts on E on the left by the formula: $T.x = Tx$ for $T \in \mathcal{K}_B(E)$ and $x \in E$. Define an A-valued inner product on E by

$$\langle x|y\rangle_A = \theta_{x,y}.$$

The proof of Proposition 10.3 imply that for $x \in E$, $||\theta_{x,x}|| = ||x||^2$. It is routine to verify that E satisfies all the axioms, except (7), of Definition 10.2. Note that for a Hilbert B-module E, the linear span of $\{\langle x|y\rangle : x, y \in E\}$ is always a two sided ideal.

Definition 10.8 *Suppose E is a Hilbert B-module. The Hilbert module E is said to be full if the closed linear span of $\{\langle x|y\rangle : x, y \in E\}$ is B.*

Proposition 10.9 *Let A and B be C^*-algebras. The following are equivalent.*

(1) The C^*-algebras A and B are Morita equivalent.

(2) There exists a full Hilbert B-module and a faithful representation $\phi : A \to \mathcal{L}_B(E)$ such that $\phi(A) = \mathcal{K}_B(E)$.

Proof. Suppose (1) holds. Let E be an A-B imprimitivity bimodule. For $a \in A$, let $\phi(a) : E \to E$ be defined by $\phi(a)(x) = ax$. In the proof of Prop. 10.3, we observed that $A \ni a \to \phi(a) \in \mathcal{L}_B(E)$ is injective. Note that for $x, y \in E$, $\phi(\langle x|y\rangle_A) = \theta_{x,y}$. Since $\{\langle x|y\rangle_A : x, y \in E\}$ is dense in A, it follows that the image of ϕ is $\mathcal{K}_B(E)$. Axiom (7) implies that E is a full Hilbert B-module.

We have already observed that if E is a full Hilbert B-module then E is a $\mathcal{K}_B(E)$-B imprimitivity bimodule. Thus (2) $\implies$ (1) is clear. $\square$

Next we show that Morita equivalence is indeed an equivalence relation on C^*-algebras.

Proposition 10.10 *Morita equivalence is an equivalence relation.*

Proof. We have already observed that A is an A-A imprimitivity bimodule. Suppose E is an A-B imprimitivity bimodule. Denote the conjugate vector space by $\overline{E}$. Then as a set $\overline{E}$ is just E. For an element $x \in E$, when we regard x as an element of $\overline{E}$, we write $j(x)$ for x. The addition and scalar multiplication are defined by

$$j(x) + j(y) = j(x + y)$$
$$\lambda . j(x) = j(\overline{\lambda} x).$$

Then $\overline{E}$ is a B-A imprimitivity bimodule where the right action of A, the left action of B and the inner products are given by

$$\begin{aligned} j(x).a &= j(a^* x) \\ b.j(x) &= j(xb^*) \\ \langle j(x)|j(y)\rangle_A &= \langle x|y\rangle_A \\ \langle j(x)|j(y)\rangle_B &= \langle x|y\rangle_B \end{aligned}$$

for $x, y \in E$ and $a \in A$, $b \in B$. Suppose E is an A-B imprimitivity bimodule and F is a B-C imprimitivity bimodule then the interior tensor product $E \otimes_B F$ is an A-C imprimitivity bimodule. $\square$

Remark 10.11 *Let E be a A-B imprimitivity bimodule and $\overline{E}$ be a conjugate B-A imprimitivity bimodule constructed in the previous proposition. Then the maps*

$$E \otimes_B \overline{E} \ni x \otimes j(y) \to \langle x|y\rangle_A \in A$$

and

$$\overline{E} \otimes_A E \ni j(x) \otimes y \to \langle x|y\rangle_B \in B$$

are isomorphisms of Hilbert modules. Thus $E \otimes_B \overline{E} \cong A$ and $\overline{E} \otimes_A E \cong B$. Thus Morita equivalent C^*-algebras have the same representation theory (See Remark 7.18).

In practice, imprimitivity bimodules are always constructed by the process of completion. The setup we usually have is as follows. Let A_0 be a dense C^*-subalgebra of A and B_0 be a dense C^*-subalgebra of B where A and B are C^*-algebras.

Definition 10.12 *A pre A_0-B_0 imprimitivity bimodule is a vector space E_0 which is a A_0-B_0 bimodule with an A_0-valued and a B_0-valued semi-definite inner products such that*

(1) the A_0-valued inner product is linear in the first variable and conjugate linear in the second variable,

(2) the B_0-valued inner product is linear in the second variable and conjugate linear in the first variable,

(3) for $x, y \in E_0$ and $a \in A_0$, $\langle ax|y\rangle_{B_0} = \langle x|a^*y\rangle_{B_0}$,

(4) for $x, y \in E_0$ and $b \in B_0$, $\langle xb|y\rangle_{A_0} = \langle x|yb^*\rangle_{A_0}$,

(5) for $x, y, z \in E_0$, $\langle x|y\rangle_{A_0} z = x\langle y|z\rangle_{B_0}$,

(6) for $x \in E_0$, $a \in A_0$ and $b \in B_0$, $\langle ax|ax\rangle_{B_0} \leq ||a||^2\langle x|x\rangle_{B_0}$ and $\langle xb|xb\rangle_{A_0} \leq ||b||^2\langle x|x\rangle_{A_0}$, and

(7) the set $\{\langle x|y\rangle_{A_0} : x, y \in E_0\}$ and the $\{\langle x|y\rangle_{B_0} : x, y \in E_0\}$ span dense ideals in A and B respectively.

Proposition 10.13 *Let E_0 be a A_0-B_0 imprimitivity bimodule. For $x \in E$,*

$$||\langle x|x\rangle_{A_0}|| = ||\langle x|x\rangle_{B_0}||.$$

Proof. Let $x \in E_0$ be given. Let $a = \langle x|x\rangle_{A_0}$. Calculate as follows to observe that

$$\begin{aligned} ||a||^2\langle x|x\rangle_{B_0} &\geq \langle ax|ax\rangle_{B_0} \\ &\geq \langle\langle x|x\rangle_{A_0}x|\langle x|x\rangle_{A_0}x\rangle_{B_0} \\ &\geq \langle x\langle x|x\rangle_{B_0}|x\langle x|x\rangle_{B_0}\rangle_{B_0} \\ &\geq \langle x|x\rangle_{B_0}^3. \end{aligned}$$

Taking norms and cancelling $||\langle x|x\rangle_{B_0}||$, we get $||\langle x|x\rangle_{A_0} \geq ||\langle x|x\rangle_{B_0}||$. A similar argument yields $||\langle x|x\rangle_{B_0}|| \geq ||\langle x|x\rangle_{A_0}||$. This completes the proof. □

Remark 10.14 *Suppose E_0 is a pre A_0-B_0 imprimitivity bimodule, first we mod out the null vectors and then complete to obtain a genuine A-B imprimitivity bimodule. The previous proposition implies that the null vectors of E_0 are the same whether we give the norm induced by the A_0-valued inner product or the B_0-valued inner product.*

We proceed towards proving Theorem 10.1 in the discrete setting. For the rest of this section, assume that G is a discrete countable group and $H \subset G$ be a subgroup. Denote the set of left cosets of H by G/H. The group G acts on G/H by left translations. Let $\alpha := \{\alpha_s\}_{s\in G}$ be the action of G on $C_0(G/H)$ induced by the left translation of G on G/H.

Let us fix notation which will be used throughout. Let $A := C_0(G/H) \rtimes G$ and $B := C^*(H)$. Denote the generating unitaries of B by $\{v_t : t \in H\}$. For $a \in C_0(G/H)$ and $s \in G$, let $a \otimes \delta_s \in C_c(G, C_0(G/H))$ be the function whose value at s is a and vanishes elsewhere. For $s \in G$, let $e_{sH} \in C_c(G/H)$ be the characteristic function at sH. Note that $\alpha_s(e_{tH}) = e_{stH}$ for $s, t \in G$. Let A_0 be the linear span of $\{e_{sH} \otimes \delta_t : s \in G, t \in G\}$ and B_0 be the linear span of $\{v_t : t \in H\}$. Then A_0 and B_0 are dense $*$-subalgebras of A and B respectively. Also note that $\{v_t : t \in H\}$ and $\{e_{rH} \otimes \delta_s : r, s \in G\}$ form a basis for B_0 and A_0 respectively.

Let $E_0 := C_c(G)$ and let $\{\epsilon_s : s \in G\}$ be the standard basis for E_0. Define a left A_0 action and a right B_0 action on E_0 by

$$(e_{rH} \otimes \delta_s).\epsilon_t := 1_{rH}(st)\epsilon_{st}$$
$$\epsilon_t.v_s = \epsilon_{ts}.$$

Define an A_0-valued sesquilinear form (by extending linearly in the first variable) and a B_0-valued sesquilinear form (by extending linearly in the second variable) by

$$\langle \epsilon_s | \epsilon_t \rangle_{B_0} = 1_H(s^{-1}t)v_{s^{-1}t}$$
$$\langle \epsilon_s | \epsilon_t \rangle_{A_0} = e_{sH} \otimes \delta_{st^{-1}}.$$

Theorem 10.1, in the discrete case, follows from the next theorem.

Theorem 10.15 *With the foregoing notation, E_0 is a pre A_0-B_0 imprimitivity bimodule.*

Proof. First we show that the sesquilinear forms defined are indeed positive semi-definite. Let us first deal with the B_0-valued sesquilinear form. Let $x := \sum_{s\in G} a_s\epsilon_s \in E_0$ be given. Let $F := \{s \in G : a_s \neq 0\}$. Then F is a finite subset of G. Define an equivalence relation on F by for $s_1, s_2 \in F$, $s_1 \sim s_2$ if and only if $s_1H = s_2H$. For $s \in F$, let $[s]$ be the equivalence class containing s. List the equivalence classes as $[s_1], [s_2], \cdots, [s_m]$. Write $[s_i] = \{s_i h_{ij} : j = 1, 2, \cdots, k_i\}$ for every $i = 1, 2, \cdots, m$.

Calculate as follows to observe that

$$\begin{aligned}
\langle x|x\rangle_{B_0} &= \sum_{i=1}^{m}\Big(\sum_{r=1}^{k_i}\overline{a_{s_ih_{ir}}}a_{s_ih_{is}}v_{h_{ir}^{-1}h_{is}}\Big)\\
&= \sum_{i=1}^{m}\Big((\sum_{j=1}^{k_i}a_{s_ih_{ij}}v_{h_{ij}})^*(\sum_{j=1}^{k_i}a_{s_ih_{ij}}v_{h_{ij}})\Big)\\
&\geq 0.
\end{aligned}$$

This shows that the B_0-valued sesquilinear form is a semidefinite inner product.

Let $x := \sum_{s\in G} a_s\epsilon_s \in E_0$ be given. Fix a non-degenerate representation of the crossed product $C_0(G/H) \rtimes G$. In other words, fix a covariant representation (π, U) of the C^*-dynamical system $(C_0(G/H), G, \alpha)$. Calculate as follows to observe that

$$\begin{aligned}
(\pi \rtimes U)(\langle x|x\rangle_{A_0}) &= \sum_{s,t\in G} a_s\overline{a_t}\pi(e_{sH})U_{st^{-1}}\\
&= \sum_{s,t\in G} a_s\overline{a_t}U_s\pi(e_H)U_t^* \quad (\text{since } U_s\pi(e_H)U_s^* = \pi(e_{sH}))\\
&= \big(\sum_{s\in G} a_sU_s\pi(e_H)\big)\big(\sum_{s\in G} a_sU_s\pi(e_H)\big)^*\\
&\geq 0.
\end{aligned}$$

Since $(\pi \rtimes U)(\langle x|x\rangle_{A_0}) \geq 0$ for every covariant representation (π, U), it follows that $\langle x|x\rangle_{A_0} \geq 0$ in A. The verifications of the axioms, except Axiom (6), of Defn. 10.12 are routine and we leave the verification to the reader.

View E_0 as a pre Hilbert B_0-module. Mod out the null vectors and complete to obtain a Hilbert B-module E. For $s \in G$ and $x := \sum_{t\in G} a_t\epsilon_t$, let $U_s(x) = \sum_{t\in G} a_t\epsilon_{st}$. Note that for $x, y \in E_0$,

$$\begin{aligned}
\langle U_sx|U_sx\rangle_{B_0} &= \langle x|x\rangle_{B_0}\\
\langle U_sx|y\rangle_{B_0} &= \langle x|U_{s^{-1}}y\rangle_{B_0}.
\end{aligned}$$

Thus there exists a unique adjointable operator on E, which again denote by U_s, such that $U_s\epsilon_t = \epsilon_{st}$.

For $x = \sum_{t\in G} a_t\epsilon_t \in E_0$, define $Px = \sum_{t\in G} 1_H(t)a_t\epsilon_t$. It is clear that $P^2x = Px$ and

$\langle Px|y\rangle = \langle x|Py\rangle$ for $x, y \in E_0$. For $x \in E_0$, calculate as follows to observe that

$$\begin{aligned}\langle x|x\rangle_{B_0} &= \langle (1-P)x + Px|(1-P)x + x\rangle_{B_0} \\ &= \langle (1-P)x|(1-P)x\rangle_{B_0} + \langle Px|Px\rangle_{B_0} \\ &\geq \langle Px|Px\rangle_{B_0}.\end{aligned}$$

The above inequality implies that there exists a unique adjointable operator, again denoted P, such that $P\epsilon_t = 1_H(t)\epsilon_t$. For $s \in G$, set $P_{sH} = U_s P U_s^*$. Note that P_{sH} is a projection. Then

$$P_{sH}(\epsilon_t) = 1_{sH}(t)\epsilon_t.$$

Hence it follows that $P_{sH}P_{tH} = 1_H(s^{-1}t)$. Making use of Proposition 2.5, we conclude that there exists a unique $*$-homomorphism $\pi : C_0(G/H) \to \mathcal{L}_B(E)$ such that $\pi(e_{sH}) = P_{sH}$. It is routine to verify that (π, U) is a covariant representation of the dynamical system $(C_0(G/H), G, \alpha)$. Also for $a \in A_0$ and $x \in E_0$, $a.x = (\pi \rtimes U)(a)x$. Calculate as follows to observe that for $a \in A_0$ and $x \in E_0$,

$$\begin{aligned}\langle ax|ax\rangle_{B_0} &= \langle (\pi \rtimes U)(a)x|(\pi \rtimes U)(a)x\rangle_B \\ &\leq ||a||^2\langle x|x\rangle_B \\ &\leq ||a||^2\langle x|x\rangle_{B_0}.\end{aligned}$$

The verification of the second half of Axiom (6) is similar and therefore relegated to an exercise. □

Exercise 10.2 *Verify the second half of Axiom* (6) *and complete the proof of the previous Theorem.*

Remark 10.16 *For examples and applications of Mackey's imprimitivity theorem, we recommend Tyrone Crisp's notes available online at www.math.ru.nl/ tcrisp.*

11 K_0 of a C^*-algebra

In the next six sections, we give a basic introduction to the subject of K-theory. We will not give complete proofs of many results and merely give a sketch. The reader interested in a detailed development should consult [3], [13] or [16].

Let A be a unital algebra over $\mathbb{C}$. Denote the set of isomorphism classes of finitely generated projective right A-modules [5] by $\mathcal{V}(A)$. Then $\mathcal{V}(A)$ is an abelian semigroup with identity. First we obtain a better description of $\mathcal{V}(A)$ in terms of idempotents. Let M be a right A-module. Recall that M is said to be finitely generated and projective if there exists a right A-module N such that $M \oplus N$ is isomorphic to A^n for some $n \geq 1$. We always think of elements of A^n as column vectors.

Exercise 11.1 *Let $m, n \geq 1$ be given. For $x \in M_{m \times n}(A)$, let $T_x : A^n \to A^m$ be defined by $T_x(v) = xv$. Show that the map*

$$M_{m \times n}(A) \ni x \to T_x \in \mathcal{L}_A(A^n, A^m)$$

is an isomorphism. Here $\mathcal{L}_A(A^n, A^m)$ denotes the abelian group of A-linear maps from A^n to A^m.

Proposition 11.1 *We have the following.*

(1) For an idempotent $e \in M_n(A)$, eA^n is a finitely generated projective A-module.

(2) Let M be a finitely generated projective A-module. Then there exists a natural number n and an idempotent $e \in M_n(A)$ such that M is isomorphic to eA^n.

(3) Let $e \in M_m(A)$ and $f \in M_n(A)$ be such that e and f are idempotents. Then eA^m and fA^n are isomorphic as A-modules if and only if there exist $x \in M_{m \times n}(A)$ and $y \in M_{n \times m}(A)$ such that $xy = e$ and $yx = f$.

Proof. Let $e \in M_n(A)$ be an idempotent. Clearly, $eA^n \oplus (1-e)A^n = A^n$. Therefore eA^n is a finitely generated projective A-module. This proves (1). Let M be a finitely generated projective A-module. Choose an A-module N such that $M \oplus N = A^n$ for some n. Let $T : A^n \to A^n$ be the map defined by $T(m \oplus n) = m$. Then T is clearly an idempotent. Hence there exists $e \in M_n(A)$ such that e is an idempotent and T is given by left multiplication by e. Note that $M = eA^n$. This proves (2).

[5] We only consider right A-modules.

Let $e \in M_m(A)$ and $f \in M_n(A)$ be such that e and f are idempotents. Suppose that eA^n and fA^m are isomorphic. Let $T : fA^n \to eA^m$ be an isomorphism and let S be the inverse of T. Decompose A^n as $A^n = fA^n \oplus (1-f)A^n$ and A^m as $A^m = eA^m \oplus (1-e)A^m$. Let $X : A^n \to A^m$ be defined by $X(u, v) = (Tu, 0)$ and $Y : A^m \to A^n$ be defined by $Y(u, v) = (Su, 0)$. Then XY is given by left multiplication by e and YX is given by left multiplication by f. Let $x \in M_{m\times n}(A)$ be the matrix corresponding to X and $y \in M_{n\times m}(A)$ be the matrix corresponding to Y. Then $xy = e$ and $yx = f$. This proves the "only if" part.

Suppose there exists $x \in M_{m\times n}(A)$ and $y \in M_{n\times m}(A)$ such that $xy = e$ and $yx = f$. Replacing x by exf and y by fye, we can assume that $exf = x$ and $fye = y$. Let $X : A^n \to A^m$ and $Y : A^m \to A^n$ be the A-linear maps that correspond to x and y respectively. Then X maps fA^n into eA^m and Y maps eA^m into fA^n. Clearly, when restricted to fA^n and eA^m, X and Y are inverses of each other. This proves the "if part". This completes the proof. $\square$

In view of Prop. 11.1, the semigroup $\mathcal{V}(A)$ can be described as follows. For $n \geq 1$, let $E_n(A)$ be the set of idempotents in $M_n(A)$. Set

$$M_\infty(A) := \bigcup_{n=1}^{\infty} M_n(A)$$
$$E_\infty(A) := \bigcup_{n=1}^{\infty} E_n(A).$$

Define an equivalence relation on $E_\infty(A)$ as follows: For $e \in E_m(A)$ and $f \in E_n(A)$, we say $e \sim f$ if there exist $x \in M_{m\times n}(A)$ and $y \in M_{n\times m}(A)$ such that $xy = e$ and $yx = f$. Then $\mathcal{V}(A) = E_\infty(A)/\sim$. Moreover the addition operation is as follows. For $e, f \in E_\infty(A)$,

$$e \oplus f = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}.$$

In a C^*-algebra, we can replace idempotents by projections and the equivalence relation then becomes Murray-von Neumann equivalence. Let A be a unital C^*-algebra. For $n \geq 1$, let $P_n(A)$ be the set of projections in $M_n(A)$. Set

$$P_\infty(A) := \bigcup_{n=1}^{\infty} P_n(A).$$

Let $p \in P_m(A)$ and $q \in P_n(A)$ be given. We say that p and q are *Murray-von Neumann* equivalent if there exists a partial isometry $u \in M_{n\times m}(A)$ such that $u^*u = p$ and $uu^* = q$.

Exercise 11.2 *Let $p \in P_m(A)$ be given. Show that p is Murray-von Neumann equivalent to* $\begin{bmatrix} p & 0 \\ 0 & 0_n \end{bmatrix}$.

Proposition 11.2 *Let A be a unital C^*-algebra.*

(1) Let $e \in E_\infty(A)$ be given. Then there exists $p \in P_\infty(A)$ such that $e \sim p$.

(2) Let $p, q \in P_\infty(A)$ be given. Then $p \sim q$ if and only if p and q are Murray-von Neumann equivalent.

Proof. Let $e \in E_n(A)$ be given. Without loss of generality, we can assume that $n = 1$. Represent A faithfully as bounded operators on a Hilbert space $\mathcal{H}$ in a unital fashion. Let p be the orthogonal projection onto $Ran(e) = Ker(1-e)$. Decompose $\mathcal{H}$ as $\mathcal{H} := Ran(p) \oplus Ker(p)$. With respect to this decomposition, e has the following matrix form

$$e := \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}.$$

Set $z := 1 + (e - e^*)(e^* - e)$. Then z is invertible in A. A simple matrix calculation implies that

$$ee^*z^{-1} = \begin{bmatrix} 1 + xx^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (1 + xx^*)^{-1} & 0 \\ 0 & (1 + x^*x)^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0. \end{bmatrix}$$

Hence $p = ee^*z^{-1}$. This implies in particular that $p \in A$. Let $x = e$ and $y = p$. Again a direct matrix calculation implies that $xy = p$ and $yx = e$. Hence $e \sim p$ and the proof of (1) is complete.

Let $p, q \in P_\infty(A)$ be given. Suppose $p \sim q$. By adding zeros along the diagonal, we can assume that p and q are of the same size. Again without loss of generality, we can assume $p, q \in A$. Let $x, y \in A$ be such that $xy = p$ and $yx = q$. Replacing x by pxq and y by qyp, if necessary, we can assume that $pxq = x$ and $qyp = y$. Consequently $y : p\mathcal{H} \to q\mathcal{H}$ and $x : q\mathcal{H} \to p\mathcal{H}$ are inverses of each other. Also y^* maps $q\mathcal{H}$ to $p\mathcal{H}$ and x^* maps $p\mathcal{H}$ to $q\mathcal{H}$. Hence $y^*y : p\mathcal{H} \to p\mathcal{H}$ is invertible. Moreover $y^*y \in pAp$. Hence there exists $r \in pAp$ such that $(y^*y)^{\frac{1}{2}}r = r(y^*y)^{\frac{1}{2}} = p$. Set $u := yr$. Note that

$y = u(y^*y)^{\frac{1}{2}}$. Clearly, $u^*u = p$ and $uu^* \leq q$. Calculate as follows to observe that

$$\begin{aligned} q &= qq^* \\ &= (yx)(x^*y) \\ &\leq ||x||^2 yy^* \\ &= ||x||^2 u(y^*y)^{\frac{1}{2}}(y^*y)^{\frac{1}{2}}y^* \\ &\leq ||x||^2||y||^2 uu^*. \end{aligned}$$

Therefore $q \leq uu^*$. Consequently, $uu^* = q$. Hence p and q are Murray-von Neumann equivalent. This completes the proof. $\Box$.

Grothendieck construction: The Grothendieck construction allows us to construct an abelian group out of the semigroup $\mathcal{V}(A)$. Let $(R, +)$ be an abelian semigroup with identity 0. Define an equivalence on $R \times R$ as follows: for $(a, b), (c, d) \in R \times R$, we say $(a, b) \sim (c, d)$ if there exists $e \in R$ such that $a + d + e = b + c + e$. Then $\sim$ is an equivalence relation on $R \times R$. Denote the set of equivalence classes by $G(R)$. Then $G(R)$ becomes an abelian group with the addition defined as

$$[(a, b)] + [(c, d)] = [(a + c, b + d)].$$

For any $a \in R$, $[(a, a)]$ represents the identity element and the inverse of $[(a, b)]$ is $[(b, a)]$. For $a \in R$, let $[a] := [(a, 0)]$. With this notation,

$$G(R) = \{[a] - [b] : a, b \in R\}.$$

Note that $[a] = [b]$ if and only if $a + c = b + c$ for some $c \in R$.

Let A be a unital algebra over $\mathbb{C}$. The Grothendieck group $G(\mathcal{V}(A))$ is denoted $K_{00}(A)$. Note that

$$K_{00}(A) = \{[p] - [q] : p, q \in E_\infty(A)\}.$$

Also $[p] = [q]$ if and only if there exists $r \in E_\infty(A)$ such that $\begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix} \sim \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}$.

Let $\phi : A \to B$ be a unital homomorphism. For $n \geq 1$, let $\phi^{(n)} : M_n(A) \to M_n(B)$ be the amplification of ϕ, i.e.

$$\phi^{(n)}((a_{ij})) = (\phi(a_{ij})).$$

To save notation, we denote $\phi^{(n)}$ again by ϕ. A moment's reflection with definitions reveal that there exists a unique homomorphism denoted $K_{00}(\phi) : K_{00}(A) \to K_{00}(B)$ such that

$$K_{00}(\phi)([p] - [q]) = [\phi(p)] - [\phi(q)].$$

In short, K_{00} is a covariant functor from the category of unital algebras to the category of abelian groups.

Remark 11.3 *Let A be a unital C^*-algebra.*

(1) Let $p, q \in A$ be projections such that $pq = 0$. Then $[p+q] = [p] + [q]$. For if we set $u := (p, q)$ then $u^*u = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ and $uu^* = p + q$.

(2) Let $p, q \in P_n(A)$ be given. Then $[p] = [q]$ in $K_{00}(A)$ if and only if there exists $m \geq 1$ such that $\begin{bmatrix} p & 0 \\ 0 & 1_m \end{bmatrix}$ is Murray-von Neumann equivalent to $\begin{bmatrix} q & 0 \\ 0 & 1_m \end{bmatrix}$

The "if part" is clear. For the "only if" part, suppose $[p] = [q]$ in $K_{00}(A)$. Then there exists $r \in P_m(A)$ such that $p \oplus r \sim q \oplus r$. Note that

$$\begin{aligned} p \oplus 1_m &\sim p \oplus (r + 1 - r) \\ &\sim p \oplus (r \oplus (1 - r)) \\ &\sim (q \oplus r) \oplus (1 - r) \\ &\sim q \oplus (r + 1 - r) \\ &\sim q \oplus 1_m. \end{aligned}$$

Exercise 11.3 *(1) Show that $K_{00}(\mathbb{C})$ is isomorphic to $\mathbb{Z}$ and $[1]$ forms a $\mathbb{Z}$-basis for $K_{00}(\mathbb{C})$.*

(2) Let $n \geq 1$. Show that $K_{00}(M_n(\mathbb{C})) = \mathbb{Z}$. Let p be a minimal projection in $M_n(\mathbb{C})$. Show that $[p]$ is a $\mathbb{Z}$-basis for $K_{00}(M_n(\mathbb{C}))$.

(3) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Show that $K_{00}(B(\mathcal{H})) = 0$.

Exercise 11.4 *Let A_1 and A_2 be unital algebras and set $A := A_1 \oplus A_2$. Show that the map $K_{00}(\pi_1) \oplus K_{00}(\pi_2) : K_{00}(A) \to K_{00}(A_1) \oplus K_{00}(A_2)$ is an isomorphism.*

Next we define K_0 for a C^*-algebra. Let A be a C^*-algebra (unital or non-unital). Set $A^+ := \{(a, \lambda) : a \in A, \lambda \in \mathbb{C}\}$. The addition and scalar multiplication on A^+ are defined co-ordinate wise. The multiplication rule is given by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

Let $\epsilon : A^+ \to \mathbb{C}$ be the map defined by $\epsilon(a, \lambda) = \lambda$. Let $s : A^+ \to A^+$ be defined by $s(a, \lambda) = (0, \lambda)$. The map s is called the "scalar map" as it remembers only the scalar part. We denote the amplifications of ϵ and s by ϵ and s itself.

Define

$$K_0(A) := Ker(K_{00}(\epsilon) : K_{00}(A^+) \to K_{00}(\mathbb{C}) = \mathbb{Z}).$$

Proposition 11.4 (The standard picture) *Let A be a C^*-algebra. Then*

$$K_0(A) = \{[p] - [s(p)] : p \in P_\infty(A^+)\}.$$

Proof. It is clear that for $p \in P_\infty(A^+)$, $[p] - [s(p)] \in K_0(A)$. Let $x \in K_0(A)$ be given. Write $x = [p] - [q]$ with p, q projections of same size say of size n. The fact that $x \in K_{00}(\epsilon)$ implies that $\epsilon(p)$ and $\epsilon(q)$ are of same rank. Choose a scalar unitary u such that $\epsilon(p) = u\epsilon(q)u^*$. Replacing q by uqu^*, we can assume that $x = [p] - [q]$ with $\epsilon(p) = \epsilon(q)$. Set $e := \begin{bmatrix} p & 0 \\ 0 & 1-q \end{bmatrix}$ and $f := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $x = [e] - [f]$. Also note that $[s(e)] = [s(p)] + [1_n - s(q)] = [1_n] = [f]$. Therefore $x = [e] - [s(e)]$. This completes the proof. $\square$

Proposition 11.5 *Let A be a unital C^*-algebra. Then $K_0(A)$ is isomorphic to $K_{00}(A)$.*

Proof. Note that the map $A^+ \ni (a, \lambda) \to (a + \lambda 1_A, \lambda) \in A \oplus \mathbb{C}$ is an isomorphism. With respect to this isomorphism, the map ϵ becomes the second projection. The result follows immediately from the previous exercise. $\square$.

K_0 **as a functor:** Let $\phi : A \to B$ be a $*$-algebra homomorphism. The map ϕ induces a map $\phi^+ : A^+ \to B^+$ which is defined as

$$\phi^+((a, \lambda)) = (\phi(a), \lambda).$$

Note that $\epsilon_B \circ \phi^+ = \epsilon_A$. Hence $K_{00}(\epsilon_B) \circ K_{00}(\phi^+) = K_{00}(\epsilon_A)$. Therefore $K_{00}(\phi^+)$ maps $K_0(A)$ to $K_0(B)$. We denote the restriction of $K_{00}(\phi^+)$ to $K_0(A)$ by $K_0(\phi)$. Thus K_0 is a functor from the category of C^*-algebras to the category of abelian groups. The functor K_0 is stable, homotopy invariant, half-exact and split-exact. We explain this in what follows.

Stability: Let A be a C^*-algebra and p be a minimal projection of $M_n(\mathbb{C})$. Let $\omega : A \to M_n(A) = A \otimes M_n(\mathbb{C})$ be defined by

$$\omega(a) := a \otimes p.$$

Then $K_0(\omega) : K_0(A) \to K_0(M_n(A))$ is an isomorphism. The reason is a matrix with entries being matrices over A is again a matrix with entries in A. We omit the proof and refer the reader to [13].

Homotopy invariance: Let A and B be C^*-algebras and $\phi, \psi : A \to B$ be $*$-homomorphisms. We say that ϕ and ψ are homotopy equivalent if there exists a family of $*$-homomorphisms $\phi_t : A \to B$ for $t \in [0,1]$ such that

(1) for $a \in A$, the map $[0,1] \ni t \to \phi_t(a) \in B$ is norm continuous, and

(2) $\phi_0 = \phi$ and $\phi_1 = \psi$.

The homotopy invariance of K_0 implies that if ϕ and ψ are two homotopy equivalent $*$-homomorphisms then $K_0(\phi) = K_0(\psi)$. A moment's thought reveals that this amounts to proving the next lemma.

Lemma 11.6 *Let $e, f \in A$ be such that e and f are projections. Suppose that $||e - f|| < 1$. Then e and f are Murray-von Neumann equivalent.*

Proof. Let $x := ef$. Note that $||x^*x - f|| = ||f(e-f)f|| < 1$. Hence x^*x is invertible in fAf. Choose $r \in fAf$ such that $r(x^*x)^{\frac{1}{2}} = (x^*x)^{\frac{1}{2}}r = f$. Set $u := xr$. Then $u^*u = f$. Since $eu = u$, it follows that $uu^* \leq e$. Represent A faithfully on a Hilbert space, say $\mathcal{H}$. Suppose that uu^* is a proper subprojection of e. Then there exists $\xi \in \mathcal{H}$ such that $e\xi = \xi \neq 0$ but $u^*\xi = 0$. Hence $rx^*\xi = 0$. Note that $x^*\xi \in Ran(f)$ and r is 1-1 on the range space of f. Hence $x^*\xi = 0$, i.e. $fe\xi = 0$. Calculate as follows to observe that

$$\begin{aligned} ||\xi|| &= ||e^2\xi - fe\xi|| \\ &= ||(e-f)e\xi|| \\ &< ||e\xi|| = ||\xi|| \end{aligned}$$

which is a contradiction. Hence $uu^* = e$. This completes the proof. $\square$

Two C^*-algebras A and B are said to be *homotopy equivalent* if there exists $*$-homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopy equivalent to the identity homomorphisms. As an example, consider $A := C[0,1]$ and $B := \mathbb{C}$. Define $\epsilon : A \to B$ by $\epsilon(f) = f(0)$ and $\sigma : B \to A$ by $\sigma(\lambda) = \lambda$. Then, clearly $\epsilon \circ \sigma$ is identity and $\sigma \circ \epsilon$ is homotopy equivalent to the identity.

Exercise 11.5 *Suppose A and B are homotopy equivalent. Show that $K_0(A)$ and $K_0(B)$ are isomorphic. Conclude that $K_0(C(X)) = \mathbb{Z}$ for a compact contractible space X.*

The next important property of K_0 is that it is half-exact and sends split exact sequences to split exact sequences.

Proposition 11.7 *Let* $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ *be a short exact sequence of* C^**-algebras. Then the sequence*

$$K_0(I) \longrightarrow K_0(A) \xrightarrow{K_0(\pi)} K_0(B)$$

is exact in the middle.

Proof. Let $x := [p] - [s(p)] \in K_0(I)$ be given. Since $\pi^+(p) = \pi^+(s(p))$, it follows that $x \in Ker(K_0(\pi))$. Let $x \in Ker(K_0(\pi))$ be given. Write $x := [p] - [s(p)]$ with p a projection in $M_n(A^+)$. Replacing p by $\begin{bmatrix} p & 0 \\ 0 & 1_m \end{bmatrix}$ for large m, we can assume that $\pi^+(p)$ and $s(\pi^+(p))$ are Murray-von Neumann equivalent.

Let v be a partial isometry in $M_n(B^+)$ be such that $v^*v = \pi^+(p)$ and $vv^* = s(\pi^+(p))$. Let $U := \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}$. Note that $U \begin{bmatrix} \pi^+(p) & 0 \\ 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} s(\pi^+(p)) & 0 \\ 0 & 0 \end{bmatrix}$. Thus, by replacing p by $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$, we can assume that $\pi^+(p)$ And $s(\pi^+(p))$ are unitarily equivalent. Let $a \in M_n(A^+)$ be a contraction such that $\pi^+(a) = U$. Set

$$V := \begin{bmatrix} a & \sqrt{1 - aa^*} \\ -\sqrt{1 - a^*a} & a^*. \end{bmatrix}$$

Then V is a unitary and $\pi^+(V) = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}$.

Note that $\pi^+(V \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} V^*)$ is a scalar matrix. This implies in particular that $q := V \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} V^*$ lies in $M_{2n}(I^+))$. Also the scalar part of q is $\begin{bmatrix} s(p) & 0 \\ 0 & 0 \end{bmatrix}$. Consequently, $x = [q] - [s(q)] \in Im(K_0(i))$ where $i : I \to A$ denotes the inclusion. This completes the proof. $\square$

Next we show that K_0 is split exact. Let

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be a short exact sequence of C^*-algebras . We say that it is split exact if there exists a $*$-homomorphism $\mu : B \to A$ such that $\pi \circ \mu = id_B$. The map μ will then be called a splitting.

Proposition 11.8 *Let* $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ *be a split exact sequence of* C^*-*algebras with the splitting given by* $\mu : B \to A$. *Then the sequence*

$$0 \longrightarrow K_0(I) \longrightarrow K_0(A) \xrightarrow{K_0(\pi)} K_0(B) \longrightarrow 0$$

is a split exact sequence of abelian groups with the splitting given by $K_0(\mu)$.

Proof. Let $i : I \to A$ be the inclusion. We have already shown the exactness at $K_0(A)$. Since $K_0(\pi) \circ K_0(\mu) = Id$, it follows that $K_0(\pi)$ is onto. The only thing that requires proof is that $K_0(i)$ is injective. To that effect, let $x := [p] - [s(p)] \in K_0(I)$ be such that $x \in Ker(K_0(i))$.

Arguing as in Prop. 11.7, we can assume that there exists a unitary $u \in M_n(A^+)$ such that $upu^* = s(p)$. Set $w := (\mu^+ \circ \pi^+(u^*))u$. Note that $\pi^+(w)$ is a scalar. Hence $w \in M_n(I^+)$. Calculate as follows to observe that

$$\begin{aligned} wpw^* &:= (\mu^+ \circ \pi^+)(u^*)upu^*(\mu^+ \circ \pi^+)(u) \\ &= (\mu^+ \circ \pi^+)(u^* s(p) u) \\ &= (\mu^+ \circ \pi^+)(p) \\ &= s(p) \quad \text{(since } p \in I^+\text{)}. \end{aligned}$$

This proves that p and $s(p)$ are Murray-von Neumann equivalent in $M_n(I^+)$. Hence $x = 0$. This completes the proof. $\square$

Exercise 11.6 *Let* A_1 *and* A_2 *be* C^*-*algebras and* $A := A_1 \oplus A_2$. *Denote the projection of* A *onto* A_i *by* π_i. *Show that the map* $K_0(\pi_1) \oplus K_0(\pi_2) : K_0(A) \to K_0(A_1) \oplus K_0(A_2)$ *is an isomorphism.*

12 K_1 of a C^*-algebra

In this section, we define another functor, denoted K_1, from the category of C^*-algebras to abelian groups. It shares the same functorial properties with K_0. This is not a coincidence as we will see later that K_1 can indeed be defined in terms of K_0. To define K_1, we work with invertible elements or unitaries.

Let A be a unital Banach algebra. Denote the set of invertible elements of $M_n(A)$ by $GL_n(A)$. Note that $GL_n(A)$ is a topological group. Denote the connected component of 1_n by $GL_n^0(A)$. Then $GL_n^0(A)$ is a normal subgroup of $GL_n(A)$. Consider the quotient group $GL_n(A)/GL_n^{(0)}(A)$. There is a natural map from $GL_n(A)/GL_n^0(A) \to GL_{n+1}(A)/GL_{n+1}^0(A)$ given by

$$x \to \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}.$$

The group $\widetilde{K}_1(A)$ is defined as the inductive limit $\varinjlim_n GL_n(A)/GL_n^0(A)$.

Exercise 12.1 *Show that $GL_n(\mathbb{C})$ is connected. Conclude that $\widetilde{K}_1(\mathbb{C}) = 0$.*

Use the previous exercise to show that for $x \in GL_n(A)$, the elements $\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ represent the same element in $GL_{n+1}(A)/GL_{n+1}^0(A)$.

Proposition 12.1 *Let A be a unital Banach algebra.*

(1) We have $\widetilde{K}_1(A) = \{[x] : x \in GL_n(A), n \geq 1\}$.

(2) For $x, y \in GL_n(A)$, $[x] = [y]$ if and only if there exists m and a path of invertibles in $GL_{n+m}(A)$ connecting $\begin{bmatrix} x & 0 \\ 0 & 1_m \end{bmatrix}$ and $\begin{bmatrix} y & 0 \\ 0 & 1_m \end{bmatrix}$.

(3) The group operation on $\widetilde{K}_1(A)$ is given by $[x] \oplus [y] := \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$.

(4) The group $\widetilde{K}_1(A)$ is abelian.

Proof. (1) and (2) are just rephrasing the definition of the inductive limit. Statements (3) and (4) follows from Exercise 12.1. $\square$

Remark 12.2 *Let A be a unital C^*-algebra. Suppose $a \in A$ is invertible. Then $u := a|a|^{-1}$ is a unitary. Note that $(a|a|^{-t})_{t\in[0,1]}$ is a path of invertible elements connecting a to u. Using this it is routine to see that in the definition of $\widetilde{K}_1(A)$, we could have taken unitaries in place of invertible elements. We usually work with unitaries in the case of C^*-algebras.*

We denote the set of unitaries in $M_n(A)$ by $\mathcal{U}_n(A)$ and the connected component of 1_n by $\mathcal{U}_n^0(A)$. For unitaries $u, v \in \mathcal{U}_n(A)$, we write $u \sim v$ if u and v represent the same element in $\mathcal{U}_n(A)/\mathcal{U}_n^0(A)$.

It is clear that $A \to \widetilde{K}_1(A)$ is a functor from the category of unital C^*-algebras to the category of abelian groups.

Exercise 12.2 *Let A_1 and A_2 be unital C^*-algebras and $A := A_1 \oplus A_2$. Denote the projection of A onto A_i by π_i. Show that the map $\widetilde{K}_1(\pi_1) \oplus \widetilde{K}_1(\pi_2) : \widetilde{K}_1(A) \to \widetilde{K}_1(A_1) \oplus \widetilde{K}_1(A_2)$ is an isomorphism.*

For any C^*-algebra A, define $K_1(A) := \widetilde{K}_1(A^+)$. For unital C^*-algebras, we have $A^+ = A \oplus \mathbb{C}$. Since $\widetilde{K}_1(\mathbb{C}) = 0$, it follows that $K_1(A) = \widetilde{K}_1(A)$. Also K_1 is a functor. If $\phi : A \to B$ is a $*$-homomorphism then there exists a unique group homomorphism $K_1(\phi) : K_1(A) \to K_1(B)$ such that

$$K_1(\phi)([u]) = [\phi^+(u)].$$

Next we discuss the functorial properties of K_1.

Stability: Let A be a C^*-algebra and p be a minimal projection in $M_n(\mathbb{C})$. Let $\omega : A \to A \otimes M_n(\mathbb{C}) = M_n(A)$ be defined by

$$\omega(a) := a \otimes p.$$

Then $K_1(\omega) : K_1(A) \to K_1(M_n(A))$ is an isomorphism. As with K_0, we omit its proof and refer the reader to [13].

Homotopy invariance: Let A and B be C^*-algebras. Suppose $\phi : A \to B$ and $\psi : A \to B$ are $*$-homomorphisms that are homotopy equivalent. Then $K_1(\phi) = K_1(\psi)$. This is obvious since homotopy invariance is built in the definition of K_1.

Lemma 12.3 *Let A be a unital C^*-algebra and $u \in A$ be a unitary. Then $u \in \mathcal{U}^0(A)$ if and only if there exists $a_1, a_2, \cdots, a_n \in A$ such that a_i's are self-adjoint and $u = e^{ia_1}e^{ia_2}\cdots e^{ia_n}$.*

Proof. For a self-adjoint element a, $(e^{ita})_{t\in[0,1]}$ is a path of unitaries connecting 1 to e^{ia}. Thus the "if part" is clear. Suppose $u \sim 1$. Let $(u_t)_{t\in[0,1]}$ be a path of unitaries such that $u_0 = 1$ and $u_1 = u$. By uniform continuity, there exists a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ such that $||u_{t_i} - u_{t_{i-1}}|| < 1$. Set $u_i := u_{t_i}$.

We claim u_1 is of the required form. Since $||u_1 - 1|| < 1$, it follows that $-1 \notin \sigma(u_1)$. Define $a := -ilog(u_1)$. Then $u_1 = e^{ia}$. This proves the claim. Note that $||u_1^*u_2 - 1|| = ||u_1 - u_2|| < 1$. Applying the above argument, we conclude that $u_1^*u_2$ is of the required form. But $u_2 = u_1u_1^*u_2$. This proves that u_2 is the required form. Proceeding this way, we see that $u_n = u$ is of the form required. This completes the proof. □

Proposition 12.4 *Let $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ be a short exact sequence of C^*-algebras. Then the sequence*

$$K_1(I) \longrightarrow K_1(A) \xrightarrow{K_1(\pi)} K_1(B)$$

is exact in the middle.

Proof. Let $i : I \to A$ be the inclusion. Let $[u] \in K_0(I)$ be given. Then $\pi^+ \circ i^+(u)$ is a scalar matrix. Consequently, $[\pi^+ \circ i^+(u)] = [1]$. Hence $Im(K_1(i)) \subset Ker(K_1(\pi))$. Let $u \in \mathcal{U}_n(A^+)$ be such that $[\pi^+(u)] = [1_n]$. Replacing u by $\begin{bmatrix} u & 0 \\ 0 & 1_m \end{bmatrix}$ for m sufficiently large, we can assume that $\pi^+(u) \sim 1_n$. Choose self-adjoint elements $b_1, b_2, \cdots, b_r \in M_n(B^+)$ such that

$$\pi^+(u) = e^{ib_1}e^{ib_2}\cdots e^{ib_r}.$$

Choose $a_i \in M_n(A^+)$ such that a_i is self-adjoint and $\pi^+(a_i) = b_i$. Set $v := e^{ia_1}e^{ia_2}\cdots e^{ia_r}$. Then $\pi^+(uv^*) = 1$. This implies that there exists $w \in \mathcal{U}_n(I^+)$ such that $uv^* = i^+(w)$. Since $[v] = 1$, it follows that $[u] = [uv^*] = K_1(i)([w])$. Hence $Im(K_1(i)) = Ker(K_1(\pi))$. This completes the proof. □

Proposition 12.5 *Let $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ be a split exact sequence of C^*-algebras with the splitting given by $\mu : B \to A$. Then the sequence*

$$0 \longrightarrow K_1(I) \longrightarrow K_1(A) \xrightarrow{K_1(\pi)} K_1(B) \longrightarrow 0$$

is a split exact sequence of abelian groups with the splitting given by $K_1(\mu)$.

Proof. Let $i : I \to A$ be the inclusion. We have already shown the exactness at $K_1(A)$. Since $K_1(\pi) \circ K_1(\mu) = Id$, it follows that $K_1(\pi)$ is onto. The only thing that requires proof is that $K_1(i)$ is injective.

Let $u \in \mathcal{U}_n(I^+)$ be such that $K_1(i)([u]) = [1_n]$. By "amplifying u", if necessary, we can assume that $i^+(u) \sim 1_n$. Let $(w_t)_{t\in[0,1]}$ be a path of unitaries in $M_n(A^+)$ such that $w_0 = 1_n$ and $w_1 = i^+(u)$. Set $v_t := (\mu^+ \circ \lambda^+)(w_t^*)w_t$. Then $\pi^+(v_t) = 1$. Hence there exists $u_t \in \mathcal{U}_n(I^+)$ such that $i^+(u_t) = w_t$. Note that $(u_t)_{t\in[0,1]}$ is a path of unitaries in $M_n(I^+)$ connecting 1_n to xu where x is a scalar matrix. Hence $[u] = [1_n]$. Therefore, $K_1(i)$ is injective. This completes the proof. $\square$

Exercise 12.3 *Let A_1 and A_2 be C^*-algebras and $A := A_1 \oplus A_2$. Denote the projection of A onto A_i by π_i. Show that the map $K_1(\pi_1) \oplus K_1(\pi_2) : K_1(A) \to K_1(A_1) \oplus K_1(A_2)$ is an isomorphism.*

13 Inductive limits and K-theory

An important property of K-theory that allows to compute the K-groups for a large class of C^*-algebras, called AF-algebras, is that it preserves direct limits. The purpose of this section is to explain this. The data that we require to define the inductive limit of C^*-algebras is as follows.

Let $(A_n)_{n\geq 1}$ be a sequence of C^*-algebras and $\phi_n : A_n \to A_{n+1}$ be a $*$-homomorphism. The above data is usually given pictorially as follows:

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \longrightarrow \cdots$$

For $m < n$, let $\phi_{n,m} : A_m \to A_n$ be defined by $\phi_{m,n} := \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m$. For $m = n$, set $\phi_{m,n} = Id$. Note that for $\ell \leq m \leq n$,

$$\phi_{n,\ell} = \phi_{n,m} \circ \phi_{m,\ell}.$$

Let $\mathcal{B} := \{(a,n) : a \in A_n, n \geq 1\}$. Define an equivalence relation on $\mathcal{B}$ as follows. For $(a,m),(b,n) \in \mathcal{B}$, we say $(a,m) \sim (b,n)$ if there exists r,s such that $m+r = s+n$ and $\phi_{m+r,m}(a) = \phi_{n+s,n}(b)$. Denote the set of equivalence classes by $\mathcal{A}_\infty$. The set $\mathcal{A}_\infty$ has a $*$-algebra structure where addition, scalar multiplication, multiplication and the $*$-structure are as follows.

$$\begin{aligned}
[(a,m)] + [(b,n)] &= [(\phi_{m+n,m}(a) + \phi_{m+n,n}(b), m+n)] \\
\lambda[(a,m)] &= [(\lambda a, m)] \\
[(a,m)][(b,n)] &= [(\phi_{m+n,m}(a)\phi_{m+n,n}(b), m+n)] \\
[(a,m)]^* &= [(a^*, m)].
\end{aligned}$$

On $\mathcal{A}_\infty$, define a C^*-seminorm as follows.

$$||[(a,m)]|| := \lim_{n\to\infty} ||\phi_{m+n,m}(a)||.$$

Mod out the null vectors and complete to get a genuine C^*-algebra which we denote by A_∞. Also A_∞ is called the inductive limit of (A_n, ϕ_n).

Let $i_n : A_n \to A_\infty$ be defined by $i_n(a) := [(a,n)]$. Note that $i_n \circ \phi_{n,m} = i_m$. This implies in particular that $i_n(A_n)$ is an increasing sequence of C^*-algebras. Moreover, the union $\bigcup_{n\geq 1} i_n(A_n)$ is dense in A_∞.

Proposition 13.1 (The universal property) *Keep the foregoing notation. Suppose B is a C^*-algebra and there exists $*$-homomorphisms $j_n : A_n \to B$ such that $j_n \circ \phi_{n,m} = j_m$. Then there exists a unique $*$-algebra homomorphism $\phi : A_\infty \to B$ such that*

$$\phi(i_n(a)) = j_n(a)$$

for $a \in A_n$. Moreover A_∞ is characterised by this property.

Proof. Left to the reader.

Remark 13.2 *Inductive limits of systems indexed by a general directed set can be defined. We have chosen to work with sequences for simplicity.*

Exercise 13.1 *Discuss inductive limits in the category of abelian groups. Formulate and prove a universal property in this context.*

The main theorem about inductive limits and K-theory is the following.

Theorem 13.3 *Let (A_n, ϕ_n) be a directed system of C^*-algebras and let A_∞ be the direct limit. Then*

$$K_i(A_\infty) = \lim_{n\to\infty}(K_i(A_n), K_i(\phi_n))$$

for $i = 0, 1$.

The soul of the proof of the above theorem relies in the following two propositions. The reader should convince herself that it is indeed so.

Proposition 13.4 *Let A be a C^*-algebra. Suppose A_n is an increasing sequence of C^*-subalgebras of A such that $\bigcup_{n=1} A_n$ is dense in A.*

(1) Let $e \in A$ be a projection. Then there exists a projection $f \in A_m$ for some m such that $e \sim f$.

(2) Let $e, f \in A_m$ be projections. Suppose $e \sim f$ in A. Then there exists n large such that $e \sim f$ in A_{m+n}.

Lemma 13.5 *Suppose A is a C^*-algebra. Let U be a non-empty open subset of $\mathbb{C}$. Then $E := \{a \in A : spec(a) \subset U\}$ is an open subset of A.*

Proof. Let C be the complement of U and F be the complement of E. We show that F is closed. Let a_n be a sequence in F such that $a_n \to a$. Then there exists $\lambda_n \in C$ such that $\lambda_n \in spec(a_n)$. Since $(||a_n||)$ is bounded, it follows that λ_n is bounded. By passing to a subsequence, we can assume that λ_n converges. Let $\lambda := \lim_n \lambda_n$. Since C is closed, $\lambda \in C$. Suppose $a - \lambda$ is invertible. Since $a_n - \lambda_n \to a - \lambda$, it follows that $a_n - \lambda_n$ is invertible for large n which is a contradiction. This forces that $\lambda \in spec(a)$. Hence $a \in F$. This proves that F is closed and hence the proof. □

Proof of Prop. 13.4. Let $\mathcal{B} := \bigcup_{n\geq 1} A_n$. Suppose $e \in A$ is a projection. Since $\mathcal{B}$ is dense, there exists $a \in \mathcal{B}$ such that $a = a^*$, $||a^2 - a|| < \frac{1}{4}$, $||a - e|| < \frac{1}{2}$ and $spec(a) \subset U := (-\frac{1}{4}, \frac{1}{4}) \cup (\frac{3}{4}, \frac{5}{4})$. Choose m such that $a \in A_m$. Let $h : U \to \mathbb{R}$ be defined by

$$h(t) := \begin{cases} 0 & \text{if } t \in (\frac{-1}{4}, \frac{1}{4}), \\ \\ 1 & \text{if } t \in (\frac{3}{4}, \frac{5}{4}). \end{cases} \tag{13.9}$$

Set $f := h(a)$. Clearly f is a projection in A_m. Note that $||a - h(a)|| < \frac{1}{2}$. Hence $||e - f|| \leq ||e - a|| + ||a - h(a)|| < 1$. By Lemma 11.6, it follows that e and f are Murray-von Neumann equivalent. This proves (1).

Let $e, f \in A_m$ be projections. Suppose that $e \sim f$ in A. Let $u \in A$ be such that $u^*u = e$ and $uu^* = f$. Choose a sequence $u_n \in \mathcal{B}$ such that $u_n \to u$. Set $v_n := fu_ne$. Then $v_n^*v_n \to e$ and $v_nv_n^* \to f$. Note that $v_n \in \mathcal{B}$. Thus, there exists $v \in \mathcal{B}$ such that $||v^*v - e|| < 1$, $||vv^* - f|| < 1$ and $fv = ve = v$. Let $n > m$ be such that $v \in A_n$.

Note that v^*v is invertible in eA_ne and vv^* is invertible in fA_nf. Let $r \in eA_ne$ and $s \in fA_nf$ be such that $r(v^*v)^{\frac{1}{2}} = e$ and $s(vv^*)^{\frac{1}{2}} = f$. Set $w := vr$. Then $w^*w = e$. We claim $w = sv$. To see this, note that $v(v^*v)^{\frac{1}{2}} = (vv^*)^{\frac{1}{2}}v$. Multiply by r on the right to deduce that $v = ve = (vv^*)^{\frac{1}{2}}vr$. Multiply on the left by s to deduce that $sv = s(vv^*)^{\frac{1}{2}}vr = fvr = vr$. This proves the claim.

Calculate as follows to observe that

$$\begin{aligned} ww^* &= svv^*s \\ &= s(vv^*)^{\frac{1}{2}}(vv^*)^{\frac{1}{2}}s \\ &= f. \end{aligned}$$

This proves that e and f are Murray-von Neumann equivalent in A_n. This completes the proof. □

Proposition 13.6 *Let A be a unital C^*-algebra. Suppose A_n is an increasing sequence of unital C^*-subalgebras of A such that $\bigcup_{n=1} A_n$ is dense in A.*

(1) Let $u \in A$ be a unitary element. Then there exists a unitary $v \in A_m$ for some m such that $u \sim v$.

(2) Let $u, v \in A_m$ be unitaries. Suppose $u \sim v$ in A. Then there exists n large such that $u \sim v$ in A_{m+n}.

Proof. Let $\mathcal{B} := \bigcup_{n=1}^{\infty} A_n$. Suppose $u \in A$ is a unitary element. Since $\mathcal{B}$ is dense, there exists $a \in \mathcal{B}$ such that $||u - a|| < 1$. Choose m such that $a \in A_m$. Note that $||1 - u^*a|| < 1$. Hence $b := u^*a$ is invertible. Moreover $(-\infty, 0]$ is disjoint from $spec(b)$. Choose a holomorphic branch, say ℓ, of the logarithm defined on $\mathbb{C}\backslash(-\infty, 0]$. Set $c := \ell(a)$ where c is defined using the holomorphic functional calculus. Then $(e^{tc})_{t\in[0,1]}$ is a path of invertibles connecting 1 to b. This implies in particular that $u \sim a$. Let $v := a|a|^{-1}$. Then $u \sim v$ and v is a unitary in A_m. This proves (1).

Let u, v be unitaries in A_m. Suppose $\widetilde{w} := (w_t)$ is a path of unitaries in A such that $w_0 = u$ and $w_1 = v$. Set

$$\widetilde{A} := C([0,1], A)$$
$$\widetilde{A_n} := C([0,1], A_n).$$

We can view $\widetilde{A}_n$ as a unital subalgebra of $\widetilde{A}$. Note that $\bigcup_{n=1}^{\infty} \widetilde{A}_n$ is dense in $\widetilde{A}$. Think of $\widetilde{w}$ as an element in $\widetilde{A}$. As in part (1), extract a path $(a_t)_{t\in[0,1]}$ of invertibles in A_n for some n with $n > m$ such that $||w_t - a_t|| < 1$ for every $t \in [0,1]$. Arguing as in (1) in A_n, we see that $u = w_0 \sim a_0$ in A_n and $v = w_1 \sim a_1$ in A_n. But $a_0 \sim a_1$ in A_n. Therefore $u \sim v$ in A_n. This completes the proof. □

Let A be a C^*-algebra. We say that A is *approximately finite dimensional*, also called an AF algebra, if there exists a sequence (A_n) of finite dimensional C^*-subalgebras of A such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A. Let $i_n : A_n \to A_{n+1}$ be the inclusion. Then $A := \lim_{n\to\infty}(A_n, i_n)$. Note that if A is a finite dimensional algebra then A is isomorphic to $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots M_{n_r}(\mathbb{C})$. Consequently, $K_0(A) = \mathbb{Z}^r$ and $K_1(A) = 0$. Since K_i preserves inductive limits, in principle, it is possible to compute the K-groups of an AF-algebra. In particular, $K_1(A) = 0$ for any AF-algebra. The reader should do the following K-group computation.

(1) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Denote the algebra of compact operators by $\mathcal{K}(\mathcal{H})$. Then $\mathcal{K}(\mathcal{H})$ is AF and $K_0(\mathcal{K}(\mathcal{H})) = \mathbb{Z}$. Moreover, if p is a minimal projection in $\mathcal{K}(\mathcal{H})$ then $[p]$ is a $\mathbb{Z}$-basis for $K_0(\mathcal{K}(\mathcal{H}))$.

(2) Set $A_n := M_{2^n}(\mathbb{C})$. Let $\phi_n : A_n \to A_{n+1}$ be defined by $\phi_n(A) := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. The inductive limit $A_\infty := \lim_{n\to\infty}(A_n, \phi_n)$ is called the CAR algebra. Then

$$K_0(A_\infty) = \mathbb{Z}[\frac{1}{2}] := \Big\{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\Big\}.$$

(3) Let $X := \{0,1\}^{\mathbb{N}}$ be the Cantor set. Then $C(X)$ is an AF-algebra. Its K-group is given by $K_0(C(X)) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$.

Exercise 13.2 *Prove the following stability result for K-theory. Let $\mathcal{K}$ be the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. Suppose p is a minimal projection in $\mathcal{K}$ and A is a C^*-algebra. Let $\omega : A \to \mathcal{K} \otimes A$ be defined by*

$$\omega(a) := p \otimes a.$$

Prove that $K_i(\omega)$ is an isomorphism.

Remark 13.7 *One of the first significant results in the subject is the classification of AF-algebras in terms of its K-theory. This was due to Elliot. Elliot's theorem asserts roughly that two AF-algebras are isomorphic if and only their K-theoretic invariants are the same. For a precise statement, we refer the reader to [7].*

14 Six term exact sequence

An important computational tool that enables us to calculate the K-groups explicitly is the six term exact sequence. We omit the proof altogether and merely explain the consequences. Let

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras. Then there exists maps $\partial : K_1(B) \to K_0(I)$, called the *index map*, and $\sigma : K_0(B) \to K_1(I)$ which makes the following six term sequence exact.

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\sigma} \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

Moreover the maps ∂ and σ are "natural".

The construction of the index map ∂, though tedious, is not that difficult. It is explicitly described below. Let $[u] \in K_1(B)$ be given where u is a unitary in $M_n(B^+)$. Choose a unitary $V \in M_{2n}(A^+)$ such that $\pi^+(V) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$. The justification for the existence of such a unitary is given in Prop. 11.7. Then

$$\partial([u]) = \left[V \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} V^* \right] - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \right].$$

The map ∂ defined above is well defined, i.e. it is independent of the various choices made and makes the diagram exact at $K_1(B)$ and $K_0(I)$. The construction of σ is more difficult and requires Bott periodicity.

The following is often used in applications.

Proposition 14.1 *Let $0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ be a short exact sequence of C^*-algebras. Let $u \in M_n(B^+)$ be a unitary. Suppose there exists a partial isometry $v \in M_n(A^+)$ such that $\pi^+(v) = u$. Then $\partial([u]) = [1_n - v^*v] - [1_n - vv^*]$.*

Proof. Let $V := \begin{bmatrix} v & 1 - vv^* \\ 1 - v^*vv^* & v^* \end{bmatrix}$. Then V is a unitary "lift" of $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$. Clearly,

$V \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} vv^* & 0 \\ 0 & 1 - v^*v \end{bmatrix}$. Calculate as follows to observe that

$$\begin{aligned}
\partial([u]) &= \left[\begin{pmatrix} vv^* & 0 \\ 0 & 1 - v^*v \end{pmatrix} \right] - \left[\begin{pmatrix} 1 - vv^* + vv^* & 0 \\ 0 & 0 \end{pmatrix} \right] \\
&= [vv^*] + [1 - v^*v] - [1 - vv^*] - [vv^*] \\
&= [1 - v^*v] - [1 - vv^*].
\end{aligned}$$

This completes the proof. □

As a first application, we deduce that K_1 can be defined in terms of K_0 and K_0 can be defined in terms of K_1. We say a C^*-algebra B is *contractible* if the identity homomorphism is homotopy equivalent to the zero map. If B is contractible then $K_0(B) = K_1(B) = 0$.

Let A be a C^*-algebra. Denote the C^*-algebra of continuous A-valued functions on $[0,1]$ by $C([0,1], A)$. The norm here is the supremum norm. Set

$$\begin{aligned}
CA &:= \{f \in C([0,1], A) : f(0) = 0\} \\
SA &:= \{f \in CA : f(1) = 0\}.
\end{aligned}$$

The C^*-algebra CA is called the *cone over* A and SA is called the *suspension over* A. Note that $A \to CA$ and $A \to SA$ are functors from the category of C^*-algebras to C^*-algebras.

Lemma 14.2 *The cone CA is contractible. Hence $K_0(CA) = 0$ and $K_1(CA) = 0$.*

Proof. For $t \in [0,1]$, let $\epsilon_t : CA \to CA$ be defined by $\epsilon_t(f)(s) = f(st)$. Then $(\epsilon_t)_{t \in [0,1]}$ is a homotopy of $*$-homomorphisms connecting the zero map with the identity map. This completes the proof. □.

Corollary 14.3 *For any C^*-algebra A, $K_1(A) \cong K_0(SA)$ and $K_0(A) \cong K_1(SA)$.*

Proof. Let $\epsilon : CA \to A$ be defined by $\epsilon(f) = f(1)$. Then we have the following exact sequence

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{\epsilon} A \longrightarrow 0.$$

The conclusion is immediate if we apply the six term exact sequence and the fact that $K_0(CA) = K_1(CA) = 0$. □

Remark 14.4 *Actually, we have cheated a lot. In fact, the isomorphism $K_0(A) \cong K_1(SA)$ is needed apriori to define the map σ in the six term sequence. After first proving this, σ is defined as the composite of the following maps*

$$K_0(B) \cong K_1(SB) \xrightarrow{\partial} K_0(SI) \cong K_1(I).$$

The isomorphism $K_0(A) \cong K_1(SA)$ is called the Bott periodicity in K-theory. We will discuss Cuntz' proof of it in the next two sections.

Exercise 14.1 *(1) Note that for $A = \mathbb{C}$, $SA := C_0(\mathbb{R})$. Conclude that $K_0(C_0(\mathbb{R})) = 0$ and $K_1(C_0(\mathbb{R})) = \mathbb{Z}$.*

(2) Let $\epsilon : C(\mathbb{T}) \to \mathbb{C}$ be the evaluation map at 1. Then the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}) \longrightarrow C(\mathbb{T}) \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

is split exact. Conclude that $K_0(C(\mathbb{T})) \cong \mathbb{Z}$ and $K_1(C(\mathbb{T})) \cong \mathbb{Z}$. Show that [1] forms a $\mathbb{Z}$-basis for $C(\mathbb{T})$.

(3) Compute the K-groups of $C(S^2)$ where S^2 is the unit sphere in $\mathbb{R}^3$.

As an application of the six term sequence, we compute the K-groups of the Toeplitz algebra. Recall that the Toeplitz algebra $\mathcal{T}$ is the universal C^*-algebra generated by an isometry v. Let $z \in C(\mathbb{T})$ be the generating unitary. Then there exists a unique surjective $*$-homomorphism $\pi : \mathcal{T} \to C(\mathbb{T})$ such that $\pi(v) = z$. Also, the kernel of π is isomorphic to the C^*-algebra of compact operators, denoted $\mathcal{K}$, on a separable infinite dimensional Hilbert space. Let $i : \mathcal{K} \to \mathcal{T}$ be the inclusion. We apply the six term sequence to the following exact sequence.

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0$$

Consider the six term exact sequence

$$\begin{tikzcd}
K_0(\mathcal{K}) \arrow[r] & K_0(\mathcal{T}) \arrow[r] & K_0(C(\mathbb{T})) \arrow[d, "\sigma"] \\
K_1(C(\mathbb{T})) \arrow[u, "\partial"] & K_1(\mathcal{T}) \arrow[l] & K_1(\mathcal{K}) \arrow[l]
\end{tikzcd}$$

We claim $\partial([z]) = -[p]$ where p is a rank one projection in $\mathcal{K}$. Since $\pi(v) = z$ and v is an isometry, it follows from Prop. 14.1 that $\partial([z]) = [1 - v^*v] - [1 - vv^*] = -[p]$. Note that $[p]$ is a $\mathbb{Z}$-basis for $K_0(\mathcal{K})$. Also, we know that $K_1(C(\mathbb{T})) \cong \mathbb{Z}$. Hence $[z]$

is a $\mathbb{Z}$-basis for $K_1(\mathbb{C}(\mathbb{T}))$. Therefore, ∂ is an isomorphism. Consequently, $K_0(i) = 0$. This implies that $Ker(K_0(\pi)) = 0$. Note that $K_1(\mathcal{K}) = 0$. This implies that σ is the zero map. Hence $Im(K_0(\pi)) = K_0(C(\mathbb{T}))$. Consequently, $K_0(\pi)$ is an isomorphism. Therefore $K_0(\mathcal{T})$ is isomorphic to $\mathbb{Z}$ and [1] is a $\mathbb{Z}$-basis for $K_0(\mathcal{T})$. Note that $K_1(\mathcal{K}) = 0$ and $Im(K_1(\pi)) = Ker(\partial) = 0$. Thus we have the short exact sequence

$$0 \longrightarrow K_1(\mathcal{T}) \longrightarrow 0.$$

As a consequence, $K_1(\mathcal{T}) = 0$.

15 Quasi-homomorphisms

We conclude these notes by discussing Cuntz' proof of Bott periodicity. An important technical tool that we need is the notion of quasi-homomorphisms. This also offers a first glimpse of KK-theory. The notion of quasi-homomorphisms is due to Cuntz. We know that homomorphisms between C^*-algebras induce maps at the K-theory level. The important observation due to Cuntz is that quasi-homorphisms, a sort of a generalised morphism between C^*-algebras, too induce maps at the K-theory level.

Let A and J be C^*-algebras. By a quasi-homomorphism from $A \to J$, we mean the following data. There exists a C^*-algebra $\mathcal{E}$ which contains J as an ideal and two $*$-homomorphisms $\phi_+, \phi_- : A \to \mathcal{E}$ such that for $a \in A$, $\phi_+(a) - \phi_-(a) \in J$. We simply say that

$$\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$$

is a quasi-homomorphism from A to J to mean the above data. Strictly speaking, there exists an embedding $i : J \to \mathcal{E}$ such that $i(J)$ is an ideal in $\mathcal{E}$. As usual, we suppress the embedding to be economical with notation.

Example 15.1 *Suppose $\sigma : A \to J$ is a homomorphism. Then $(\sigma, 0) : A \to J \trianglerighteq J$ is a quasi-homomorphism. More generally, suppose $\sigma_1, \sigma_2 : A \to J$ are homomorphisms. Then $\sigma := (\sigma_1, \sigma_2) : A \to J \trianglerighteq J$ is a quasi-homomorphism.*

Let $\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$ be a quasi-homomorphism. Set

$$A_\phi := \{(a, x) \in A \oplus \mathcal{E} : \phi_+(a) \equiv x \mod J\}.$$

Let $\pi : A_\phi \to A$ be defined by $\pi(a, x) = a$. Then $Ker(\pi) := \{(0, x) : x \in J\}$ which we identify with J. Let $j : J \to A_\phi$ be the embedding $j(x) = (0, x)$. Define $\widetilde{\phi_+} : A \to A_\phi$ and $\widetilde{\phi_-} : A \to A_\phi$ by

$$\widetilde{\phi_+}(a) = (a, \phi_+(a))$$
$$\widetilde{\phi_-}(a) = (a, \phi_-(a)).$$

Then we have the following split exact sequence of C^*-algebras with the splitting given by either $\widetilde{\phi_+}$ or $\widetilde{\phi_-}$.

$$0 \longrightarrow J \longrightarrow A_\phi \xrightarrow{\pi} A \longrightarrow 0$$

Hence $K_i(j)$ is injective. Note that $K_i(\widetilde{\phi_+}) - K_i(\widetilde{\phi_-}) \in Ker(K_i(\pi)) = Im(K_i(j))$. For $i = 0, 1$, we define $\widehat{K_i}(\phi) : K_i(A) \to K_i(J)$ by the formula

$$\widehat{K_i}(\phi) := K_i(j)^{-1} \circ \Big(K_i(\widetilde{\phi_+}) - K_i(\widetilde{\phi_-})\Big).$$

Next we derive a few basic properties about quasi-homomorphisms.

Proposition 15.2 *Let $\sigma := (\sigma_1, \sigma_2) : A \to J \trianglerighteq J$ be a quasi-homomorphism. Then $\widehat{K_i}(\sigma) = K_i(\sigma_1) - K_i(\sigma_2)$.*

Proof. Note that $A_\sigma = A \oplus J$. Then we can identify $K_i(A_\sigma)$ with $K_i(A) \oplus K_i(J)$. Once this identification is made, $K_i(j)^{-1}$ on $Im(K_i(j))$ is nothing but $K_i(pr_2)$ where $pr_2 : A \oplus J \to J$ is the second projection. The conclusion is now obvious. □

In view of the above proposition, for a quasi-homomorphism ϕ, we simply denote $\widehat{K}_i(\phi)$ by $K_i(\phi)$. Next we discuss how to precompose a quasi-homomorphism with a homomorphism. Let $\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$ be a quasi-homomorphism. Suppose $\epsilon : B \to A$ is a homomorphism. Then $\psi := (\psi_+, \psi_-) : B \to \mathcal{E} \trianglerighteq J$ is a quasi-homomorphism where $\psi_+ = \phi_+ \circ \epsilon$ and $\psi_- = \phi_- \circ \epsilon$.

Proposition 15.3 *With the foregoing notation, we have $K_i(\psi) = K_i(\phi) \circ K_i(\epsilon)$.*

Proof. Let $j_B : J \to B_\psi$ and $j_A : J \to A_\phi$ be the embeddings. Define $\eta : B_\psi \to A_\phi$ by

$$\eta(b, x) := (\epsilon(b), x).$$

Note that $\eta \circ j_B = j_A$, $\eta \circ \widetilde{\psi_+} = \widetilde{\phi_+} \circ \epsilon$ and $\eta \circ \widetilde{\psi_-} = \widetilde{\phi_-} \circ \epsilon$. Let $y \in K_i(B)$ given. Choose $x \in K_i(J)$ such that $K_i(j_B)x = (K_i(\widetilde{\psi}_+) - K_i(\widetilde{\psi}_-))y$. Then $K_i(\psi)y = x$.

To show that $K_i(\phi) \circ K_i(\epsilon)y = x$, it suffices to show that $K_i(j_A)x = (K_i(\widetilde{\phi}_+) - K_i(\widetilde{\phi}_-))K_i(\epsilon)y$. Calculate as follows to observe that

$$\begin{aligned}
K_i(j_A)(x) &= K_i(\eta)K_i(j_B)x \\
&= K_i(\eta)K_i(\widetilde{\psi}_+)y - K_i(\eta)K_i(\widetilde{\psi}_-)y \\
&= K_i(\widetilde{\phi}_+ \circ \epsilon)y - K_i(\widetilde{\phi}_- \circ \epsilon)y \\
&= (K_i(\widetilde{\phi}_+) - K_i(\widetilde{\phi}_-))K_i(\epsilon)y.
\end{aligned}$$

This completes the proof. □

Post composing a quasi-homomorphism with a homomorphism is a bit tricky. The data we need is the following. Suppose $\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$ is a quasi-homomorphism. Let $\epsilon' : J \to J'$ be a $*$-homomorphism. To define $\epsilon' \circ \phi$, we need an extension of ϵ'. Suppose there exists a C^*-algebra $\mathcal{E}'$ containing J' as an ideal such that ϵ' extends to a map from $\mathcal{E} \to \mathcal{E}'$. Denote an extension again by ϵ'. Set $\psi_+ := \epsilon' \circ \phi_+$ and $\psi_- := \epsilon' \circ \phi_-$. Then $\psi := (\psi_+, \psi_-) : A \to \mathcal{E}' \trianglerighteq J'$ is a quasi-homomorphism.

Proposition 15.4 *With the foregoing notation, we have $K_i(\psi) = K_i(\epsilon') \circ K_i(\phi)$.*

Proof. Let $j : J \to A_\phi$ and $j' : J' \to A_\psi$ be the embeddings. Let $\eta : A_\phi \to A_\psi$ be defined by $\eta(a,x) := (a, \epsilon'(x))$. Then $\eta \circ \widetilde{\phi_+} = \widetilde{\psi_+}$ and $\eta \circ \widetilde{\phi_-} = \widetilde{\psi_-}$. Also $\eta \circ j = j' \circ \epsilon'$.

Let $y \in K_i(A)$ be given. Choose $x \in K_i(J)$ such that $K_i(\widetilde{\phi_+})y - K_i(\widetilde{\phi_-})y = K_i(j)x$. To prove $K_i(\psi)y = K_i(\epsilon')K_i(\phi)y$, it suffices to show that $K_i(j')K_i(\epsilon')x = K_i(\widetilde{\psi_+})y - K_i(\widetilde{\psi_-})y$. Calculate as follows to observe that

$$\begin{aligned} K_i(j')K_i(\epsilon')x &= K_i(\eta)K_i(j)x \\ &= K_i(\eta)(K_i(\widetilde{\phi_+})y - K_i(\eta)K_i(\widetilde{\phi_-})y \\ &= K_i(\widetilde{\psi_+})y - K_i(\widetilde{\psi_-})y. \end{aligned}$$

This completes the proof. $\square$

For $t \in [0,1]$, let $\phi^t := (\phi^t_+, \phi^t_-) : A \to \mathcal{E} \trianglerighteq J$ be a family of quasi-homomorphisms. We say that $(\phi^t)_{t\in[0,1]}$ is a homotopy if $(\phi^t_+)_{t\in[0,1]}$ and $(\phi^t_-)_{t\in[0,1]}$ are homotopy of $*$-homomorphisms.

Proposition 15.5 *Let $\phi^t := (\phi^t_+, \phi^t_-) : A \to \mathcal{E} \trianglerighteq J$ be a homotopy of $*$-homomorphisms. Then $K_i(\phi^t)$ is independent of t.*

Proof. Let $\widetilde{\mathcal{E}} := C([0,1], \mathcal{E})$ and $\widetilde{J} := C([0,1], J)$. Define $\widetilde{\phi_+} : A \to \widetilde{\mathcal{E}}$ by the formula

$$\widetilde{\phi_+}(a)(t) := \phi^t_+(a).$$

Similarly define $\widetilde{\phi_-}$. Then $\widetilde{\phi} := (\widetilde{\phi_+}, \widetilde{\phi_-}) : A \to \widetilde{\mathcal{E}} \trianglerighteq \widetilde{J}$ is a quasi-homomorphism. For $t \in [0,1]$, let $\epsilon_t : \widetilde{\mathcal{E}} \to \mathcal{E}$ be the evaluation at t. By Prop. 15.4, $K_i(\phi^t) = K_i(\epsilon_t) \circ K_i(\widetilde{\phi})$. However, $K_i(\epsilon_t)$ is constant by the homotopy invariance of K-theory. Hence the proof. $\square$

Next we discuss the additive property of K-theory. Let $\phi, \psi : A \to B$ be homomorphisms. We say ϕ and ψ are othogonal and write $\phi \perp \psi$ if for $x, y \in A$, $\phi(x)\psi(y) = 0$. Note that if $\phi \perp \psi$ then $\phi + \psi : A \to B$ is a $*$-homomorphism. Let $\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$ and $\psi := (\psi_+, \psi_-) : A \to \mathcal{E} \trianglerighteq J$ be two quasi-homomorphisms. We say that ϕ and ψ are orthogonal if $\phi_+ \perp \psi_+$ and $\phi_- \perp \psi_-$. If ϕ and ψ are orthogonal, then clearly $\phi + \psi := (\phi_+ + \psi_+, \phi_- \circ \psi_-) : A \to \mathcal{E} \trianglerighteq J$ is a quasi-homomorphism.

Proposition 15.6 *Suppose $\phi := (\phi_+, \phi_-) : A \to \mathcal{E} \trianglerighteq J$ and $\psi := (\psi_+, \psi_-) : A \to \mathcal{E} \trianglerighteq J$ are orthogonal quasi-homomorphisms. Then $K_i(\phi + \psi) = K_i(\phi) + K_i(\psi)$.*

Lemma 15.7 *Let A be a C^*-algebra. Let $i_1, i_2 \to A \to A \oplus A$ be defined by $i_1(a) = (a, 0)$ and $i_2(a) = (0, a)$. Then $K_i(i_1 + i_2) = K_i(i_1) + K_i(i_2)$.*

Proof. Let $\pi_1, \pi_2 : A \oplus A \to A$ be the first and the second projections respectively. We know that $K_i(\pi_1) \oplus K_i(\pi_2) : K_i(A \oplus A) \to K_i(A) \oplus K_i(A)$ is an isomorphism. To verify the equality $K_i(i_1+i_2) = K_i(i_1)+K_i(i_2)$, it suffices to verify the following two equalities.

$$\begin{aligned}
K_i(\pi_1) \circ K_i(i_1 + i_2) &= K_i(\pi_1) \circ K_i(i_1) + K_i(\pi_1) \circ K_i(i_2) \\
K_i(\pi_2) \circ K_i(i_1 + i_2) &= K_i(\pi_2) \circ K_i(i_1) + K_i(\pi_2) \circ K_i(i_2)
\end{aligned}$$

This verification is obvious. $\square$.

Proof of Prop. 15.6: Let $\Sigma_+ : A \oplus A \to \mathcal{E} \trianglerighteq J$ be defined by $\Sigma_+(a, b) = \phi_+(a)+\psi_+(b)$. Similarly define Σ_-. Then $\Sigma := (\Sigma_+, \Sigma_-) : A \to \mathcal{E} \trianglerighteq J$ is a quasi-homomorphism. Let $\Delta : A \to A \oplus A$ be defined by $\Delta(a) = (a, a)$. Then $\Delta = i_1 + i_2$. Note that $\phi + \psi := \Sigma \circ \Delta$. Clearly $\phi = \Sigma \circ i_1$ and $\psi = \Sigma \circ i_2$. Calculate as follows to observe that

$$\begin{aligned}
K_i(\phi + \psi) &= K_i(\Sigma) \circ K_i(\Delta) \\
&= K_i(\Sigma) \circ (K_i(i_1) + K_i(i_2)) \\
&= K_i(\Sigma) \circ K_i(i_1) + K_i(\Sigma) \circ K_i(i_2) \\
&= K_i(\Sigma \circ i_1) + K_i(\Sigma \circ i_2) \\
&= K_i(\phi) + K_i(\psi).
\end{aligned}$$

This completes the proof. $\square$

16 Bott periodicity

In this section, we discuss Cuntz' proof of Bott periodicity. The main result in Cuntz' proof is to first compute the K-theory of the Toeplitz algebra. We had already computed the K-groups of the Toeplitz algebra assuming Bott periodicity. Here we compute it without this assumption. Recall that the Toeplitz algebra $\mathcal{T}$ is the universal C^*-algebra generated by a single isometry v.

We need to use tensor products of C^*-algebras, a delicate topic, in what follows. The reader should consult [4] for a detailed treatment. We ask the reader to accept the statements made here in good faith. Let A_1 and A_2 be C^*-algebras. Consider the algebraic tensor product $A_1 \otimes_{alg} A_2$. Then $A_1 \otimes_{alg} A_2$ is a $*$-algebra where the multiplication and $*$-structure are given by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1b_1 \otimes a_2b_2$$
$$(a \otimes b)^* = a^* \otimes b^*.$$

Let $|| \ ||$ be a C^*-norm on $A_1 \otimes_{alg} A_2$. The norm $|| \ ||$ is said to be a cross-norm on $A_1 \otimes_{alg} A_2$ if $||a \otimes b|| = ||a||||b||$. It is true that there exists C^*-algebras A_1 and A_2 such that $A_1 \otimes_{alg} A_2$ admits more than one C^* cross norm.

Definition 16.1 *A C^*-algebra A is called nuclear if the following holds. For every C^*-algebra B, there is only one C^* cross norm on the algebraic tensor product $A \otimes_{alg} B$. If A is nuclear then $A \otimes B$ denotes the completion of $A \otimes_{alg} B$ with respect to any C^* cross norm*

Exercise 16.1 *Show that $M_n(\mathbb{C})$ is nuclear.*

Spatial tensor product: It is always possible to define a C^*-cross norm as follows. Let A_1 and A_2 be two C^*-algebras. Let $\pi_1 : A_1 \to B(\mathcal{H}_1)$ and $\pi_2 : A_2 \to B(\mathcal{H}_2)$ be faithful representations. Define $\pi_1 \otimes \pi_2 : A_1 \otimes_{alg} A_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the equation

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) := \pi_1(a_1) \otimes \pi_2(a_2).$$

Then $\pi_1 \otimes \pi_2$ is a $*$-homomorphism and is injective. For $x \in A_1 \otimes_{alg} A_2$, let $||x|| := ||\pi_1 \otimes \pi_2(x)||$. Then $|| \ ||$ is a norm on $A_1 \otimes_{alg} A_2$. It is a non-trivial fact that $|| \ ||$ is independent of the chosen faithful representations π_1 and π_2. This norm on $A_1 \otimes_{alg} A_2$ is called the spatial norm and the completion of $A_1 \otimes_{alg} A_2$ is called *the spatial tensor product.* The reader can assume that the tensor product of C^*-algebras that we consider is always the spatial one without much loss.

Exercise 16.2 *Let D be a C^*-algebra. Show that the map $A \to A \otimes D$ is a functor from the category of C^*-algebras to the category of C^*-algebras. Here the tensor product is the spatial one.*

We need the following facts regarding nuclear C^*-algebras and tensor products.

1. Commutative C^*-algebras are nuclear.
2. Inductive limits of nuclear C^*-algebras are nuclear.
3. Let $0 \to I \to A \to B \to 0$ be a short exact sequence. If I and B are nuclear then A is nuclear.
4. Let $0 \to I \to A \to B \to 0$ be a short exact sequence of nuclear C^*-algebras. If D is a C^*-algebra then the sequence

$$0 \to D \otimes I \to D \otimes A \to D \otimes B \to 0$$

is exact.

Exercise 16.3 *Use the above facts to conclude that the Toeplitz algebra is nuclear.*

Exercise 16.4 *Let X be locally compact Hausdorff topological space and A be a C^*-algebra. Assume that $C_0(X)$ is nuclear. Use this assumption to show that $C_0(X) \otimes A \cong C_0(X, A)$.*

Hint: The map $C_0(X) \otimes_{alg} A \ni f \otimes a \to f.a \in C_0(X, A)$ is an embedding. Here $f.a$ stands for the map which sends x to $f(x)a$.

Let us return to the discussion on Bott periodicity. Let $q : \mathcal{T} \to \mathbb{C}$ be defined by $q(v) = 1$ and $j : \mathbb{C} \to \mathcal{T}$ be defined by $j(1) = 1$. The map q exists by Coburn's theorem. We claim that $K_i(q) : K_i(\mathcal{T}) \to K_i(\mathbb{C})$ is an isomorphism with inverse given by $K_i(j)$. Since $q \circ j = id$, it follows that $K_i(q) \circ K_i(j) = id$.

Let $p = 1 - vv^*$. Let $\omega : \mathcal{T} \to \mathcal{K} \otimes \mathcal{T}$ be defined by $\omega(x) = p \otimes x$. Since $K_i(\omega)$ is an isomorphism, to show that $K_i(j) \circ K_i(q) = Id$, it suffices to show that $K_i(\omega) \circ K_i(j \circ q) = K_i(\omega)$. Let $\sigma_1 := \omega \circ j \circ q$ and $\sigma_2 = \omega$. Then

$$\sigma_1(v) = p \otimes 1$$
$$\sigma_1(v) = p \otimes v.$$

We need to show that $K_i(\sigma_1) = K_i(\sigma_2)$.

Exercise 16.5 *Prove the following version of Coburn's theorem. Let A be a C^*-algebra. Suppose w is a partial isometry in A such that $ww^* \leq w^*w$. Then there exists a unique $*$-homomorphism $\sigma : \mathcal{T} \to A$ such that $\sigma(v) = w$.*

Hint: Set $p := w^*w$ and consider pAp.

Keep the notation preceeding the above exercise.

Theorem 16.2 *We have $K_i(\sigma_1) = K_i(\sigma_2)$. Therefore, the map $K_i(q) : K_i(\mathcal{T}) \to K_i(\mathbb{C})$ is an isomorphism.*

Proof. By Coburn's theorem, there exists a $*$-homomorphism $\epsilon : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ such that $\epsilon(v) = v(1-p) \otimes 1$. Note that

$$\sigma := (\sigma_1, \sigma_2) : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T} \trianglerighteq \mathcal{K} \otimes \mathcal{T}$$

and

$$\widetilde{\epsilon} := (\epsilon, \epsilon) : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T} \trianglerighteq \mathcal{K} \otimes \mathcal{T}$$

are quasi-homomorphisms. Moreover $\sigma \perp \widetilde{\epsilon}$. By the properties of quasi-homomorphisms discussed in the previous section, it follows that $K_i(\sigma + \widetilde{\epsilon}) = K_i(\sigma) + K_i(\widetilde{\epsilon}) = K_i(\sigma_1) - K_i(\sigma_2)$. We will be done if we show that $K_i(\sigma + \widetilde{\epsilon}) = 0$.

Let

$$\begin{aligned} v_t : &= cos(\frac{\pi}{2}t)(p \otimes 1) + sin(\frac{\pi}{2}t)(vp \otimes 1) + v(1-p) \otimes 1 \\ w_t : &= cos(\frac{\pi}{2}t)(p \otimes v) + sin(\frac{\pi}{2}t)(vp \otimes 1) + v(1-p) \otimes 1 \end{aligned}$$

Note that v_t and w_t are isometries in $\mathcal{T} \otimes \mathcal{T}$. For every $t \in [0,1]$, by Coburn's theorem, there exists $*$-homomorphisms $\sigma_+^{(t)} : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ and $\sigma_-^{(t)} : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ such that

$$\begin{aligned} \sigma_+^{(t)}(v) &= v_t \\ \sigma_-^{(t)}(v) &= w_t. \end{aligned}$$

Clearly $\sigma^{(t)} := (\sigma_+^{(t)}, \sigma_-^{(t)}) : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T} \trianglerighteq \mathcal{K} \otimes \mathcal{T}$ is a homotopy of quasi-homorphisms. Note that $\sigma^{(0)} := \sigma + \widetilde{\epsilon}$. By the homotopy invariance, we have

$$K_i(\sigma + \widetilde{\epsilon}) = K_i(\sigma_+^{(1)}, \sigma_-^{(1)}).$$

But $\sigma_+^{(1)} = \sigma_-^{(1)}$. Hence $K_i(\sigma + \widetilde{\epsilon}) = 0$. Consequently, we have $K_i(\sigma_1) = K_i(\sigma_2)$. This completes the proof. $\square$.

Corollary 16.3 *Let $\mathcal{T}_0 := Ker(q)$. Then $K_i(\mathcal{T}_0) = 0$.*

Proof. Note that the short exact exact sequence

$$0 \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T} \xrightarrow{q} \mathbb{C} \longrightarrow 0$$

is split exact with the splitting given by j. Since $K_i(q)$ is an isomorphism, the conclusion follows. $\square$

The next step in the proof of Bott periodicity is to establish that $K_i(\mathcal{T}_0 \otimes B) = 0$ for every C^*-algebra B. Actually, we do not have to do anything. If we go through the proofs once again, we realise that all we need to know about the functor K_i is that it is stable, homotopy invariant and sends split exact sequences to split exact sequences. The proof is applicable for any functor from the category of (nuclear) C^*-algebras to the category of abelian groups which is split exact, homotopy invariant and is stable.

Fix a C^*-algebra B. Let F be the functor from the category of nuclear C^*-algebras to the category of abelian groups defined by $F(A) = K_i(A \otimes B)$. Then F is split exact, stable and homotopy invariant. Therefore $F(\mathcal{T}_0) = 0$, i.e. $K_i(\mathcal{T}_0 \otimes B) = 0$. With this in hand, we can complete the proof of Bott periodicity.

Theorem 16.4 (Bott periodicity) *For any C^*-algebra B, we have*

$$K_0(B) \cong K_1(SB) = K_1(C_0(\mathbb{R}) \otimes B).$$

Proof. Let $\sigma : \mathcal{T} \to C(\mathbb{T})$ be the map that sends v to z. Denote by ev_1, the evaluation map from $C(\mathbb{T}) \to \mathbb{C}$ at 1. Note that $q = ev_1 \circ \sigma$. We can identify $C_0(\mathbb{R})$ with $Ker(ev_1)$. Consequently, we have the following short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_0 \longrightarrow C_0(\mathbb{R}) \longrightarrow 0.$$

Tensor the above short exact sequence to obtain the following.

$$0 \longrightarrow \mathcal{K} \otimes B \longrightarrow \mathcal{T}_0 \otimes B \longrightarrow C_0(\mathbb{R}) \otimes B \longrightarrow 0.$$

Since $K_0(\mathcal{T}_0 \otimes B) = K_1(\mathcal{T}_0 \otimes B) = 0$, it follows that the index map

$$\partial : K_1(C_0(\mathbb{R}) \otimes B) \to K_0(\mathcal{K} \otimes B) \cong K_0(B)$$

is an isomorphism. This completes the proof. $\square$

Remark 16.5 *In fact, the statement of Bott periodicity is a bit more. Bott periodicity gives an explicit map from $K_0(B) \to K_1(SB)$. We explain this for unital C^*-algebras. Let B be a unital C^*-algebra. Then*

$$M_n((SB)^+) := \{f : \mathbb{T} \to M_n(B) : f(1)_{ij} \in \mathbb{C}1_B\}.$$

For a projection $p \in M_n(B)$, let $f_p : \mathbb{T} \to M_n(B)$ be defined by

$$f_p(z) = zp + 1_n - p.$$

Then f_p is a unitary in $M_n((SB)^+)$.

Bott periodicity asserts that there exists a unique map $\beta : K_0(B) \to K_1(SB)$, called the Bott map, which is an isomorphism such that

$$\beta([p]) = [f_p].$$

If we carefully work through the proofs and unwrap all the identifications, we can prove that the Bott map is indeed an isomorphism. The reader should carry out this verification.

Remark 16.6 *Much of the material on K-theory is based on the lectures given by Cuntz during a conference held at Oberwolfach in 2014.*

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